MODULES OF COVARIANTS IN MODULAR INVARIANT THEORY

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Abstract. Let the finite group G act linearly on the vector space V over the field k of arbitrary characteristic, and let $H < G$ be a subgroup. The extension of invariant rings $k[V]^G \subset k[V]^H$ is studied using modules of covariants.

An example of our results is the following. Let W be the subgroup of G generated by the reflections in G. A classical theorem due to Serre says that if $k[V]$ is a free $k[V]^G$-module then $G = W$. We generalize this result as follows. If $k[V]^H$ is a free $k[V]^G$-module, then G is generated by H and W. Furthermore, the invariant ring $k[V]^{H\cap W}$ is free over $k[V]^W$ and is generated as an algebra by H-invariants and W-invariants.

Introduction

Let V be an n-dimensional vector space over a field k of arbitrary characteristic and let G be a finite group of linear automorphisms of V. Write $k[V]$ for the coordinate algebra of V; it is in a natural way a graded algebra. Then G also acts as a group of graded k-algebra automorphisms of $k[V]$, and the collection of G-invariant polynomials forms a graded k-algebra $k[V]^G$, the *algebra of invariants*. Let G also act linearly on another finite dimensional vector space M over the same base field. Then we get an induced G-action on the free $k[V]$-module $k[V] \otimes_k M$. The G-invariant elements of $k[V] \otimes_k M$ form a $k[V]^G$-submodule

$$k[V]^G(M) := (k[V] \otimes_k M)^G,$$

called the *module of covariants of type M*. The algebra of invariants and the modules of covariants are classical objects of study. We shall say the situation is *modular* if the characteristic of the field k is positive and divides the order of the group, and *non-modular* if it is not modular.

We shall say that $\sigma \in G$ is a *reflection* if it fixes a linear hyperplane of V point-wise. Put W for the (normal) subgroup generated by the reflections in G. This subgroup plays an important role. The *Hilbert series* $\mathcal{H}(k[V]^G(M); t)$ is defined as

$$\mathcal{H}(k[V]^G(M); t) := \sum_{i\geq 0} \dim_k (k[V]_i \otimes_k M)^G \cdot t^i.$$

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Define $\deg(k[V]^G(M))$ and $\psi(k[V]^G(M))$ by the expansion of the Hilbert series at $t = 1$:

$$\mathcal{H}(k[V]^G(M);t) = \frac{\deg(k[V]^G(M))}{(1-t)^n} + \frac{\psi(k[V]^G(M))}{(1-t)^{n-1}} + O(\frac{1}{(1-t)^{n-2}}).$$

In the non-modular situation, a lot more is known than in the modular situation. For example, the following results are true in the non-modular situation but not so in the modular situation. $k[V]^G$ is a Cohen-Macaulay algebra and $k[V]^G(M)$ is a Cohen-Macaulay module over $k[V]^G$, by Hochster-Eagon [18]. The invariant algebra $k[V]^G$ is a polynomial algebra *if and only if* $G = W$, by Chevalley-Shephard-Todd [3, Theorem 7.2.1], and all modules of covariants are free when $G = W$. There is a Molien formula for the Hilbert series $\mathcal{H}(k[V]^G(M);t)$ using only the (Brauer) characters of V and M, see e.g. [12].

There are other results that are generally true, but whose proofs are much harder in the modular case. For example, the following formula

$$|G|\psi(k[V]^G) = \sum_{U\subset V} |G_U|\psi(k[V]^{G_U}), \tag{1}$$

where the sum is over all linear hyperplanes $U \subset V$ and where G_U is the subgroup of G consisting of all elements that fix U point-wise. This follows easily from Molien's formula in the non-modular situation, but the proof by Benson and Crawley-Boevey [4] requires a lot more work in the modular situation.

If $k[V]$ is free as $k[V]^G$-module, then $G = W$. This result, first proved over the complex numbers by Shephard-Todd [30] and then in general by Serre [6, V §5, ex. 8], [3], requires again a different approach in the modular case (see also [9]).

Stanley [33] studied modules of covariants over the field of complex numbers when M is one-dimensional. For example, he shows that then $k[V]^W(M)$ is free of rank one over $k[V]^W$, and he explicitly describes a generator $F_M \otimes v \in (k[V] \otimes M)^W$. Here $F_M \in k[V]$ is a certain product of linear forms and $v \in M$ is a basis element of the one-dimensional module M. Then he gives the criterion that $k[V]^G(M)$ is free over $k[V]^G$ *if and only if* $k[V]^W(M)$ and $k[V]^G(M)$ share generators *if and only if* $k[V]^G(M)$ contains a non-zero homogeneous element of degree equal to the degree of F_M. Again, over a general field these results remain true, but require additional work, see Nakajima [23].

In this article, we extend in some sense the above mentioned results of Benson and Crawley-Boevey, Serre, Stanley and Nakajima, to general modules of covariants over arbitrary fields.

First, the following formula is a direct analogue of (1) for modules of covariants

$$|G|\psi(k[V]^G(M)) = \sum_{U\subset V} |G_U|\psi(k[V]^{G_U}(M)), \tag{2}$$

where the sum is over all linear hyperplanes $U \subset V$, see Theorem 1 and Remark 5.1. Define $r_{k[V]^G}(k[V]^G(M))$, called the *rank*, and $s_{k[V]^G}(k[V]^G(M))$, called the *s-invariant*, by

the Taylor expansion at $t = 1$ of the quotient of Hilbert series

$$\frac{\mathcal{H}(k[V]^G(M);t)}{\mathcal{H}(k[V]^G;t)} = r_{k[V]^G}(k[V]^G(M)) + s_{k[V]^G}(k[V]^G(M))(t-1) + O((t-1)^2).$$

The rank is just the dimension of M. If $k[V]^G(M)$ is a free $k[V]^G$-module with a homogeneous basis of degrees $e_1, \ldots, e_m$, then the s-invariant coincides with the sum of these degrees. Formula (2) is equivalent to the formula

$$s_{k[V]^G}(k[V]^G(M)) = \sum_{U \subset V,\ \mathrm{codim}_V U = 1} s_{k[V]^{G_U}}(k[V]^{G_U}(M)). \tag{3}$$

We show in Theorem 1 that the s-invariant only depends on the reflections in G, i.e.,

$$s_{k[V]^G}(k[V]^G(M)) = s_{k[V]^W}(k[V]^W(M)).$$

Since $k[V]^{G_U}(M)$ is always a free graded $k[V]^{G_U}$-module, by Proposition 6(i), it follows that the s-invariant of a module of covariants is always a non-negative integer. Furthermore, the s-invariant is 0 *if and only if* the subgroup W acts trivially on M, by Proposition 3.

Next, we describe a generalization of a part of Stanley–Nakajima's results. We prove in Theorem 2 that if $k[V]^G(M)$ is free over $k[V]^G$, then $k[V]^W(M)$ is also free over $k[V]^W$ and both modules of covariants share generators. More precisely, multiplication induces an isomorphism of $k[V]^W$-modules

$$k[V]^W \otimes_{k[V]^G} k[V]^G(M) \simeq k[V]^W(M).$$

Conversely, if $k[V]^W(M)$ is free over $k[V]^W$ and the modules of covariants $k[V]^W(M)$ and $k[V]^G(M)$ share generators, then $k[V]^G(M)$ is free over $k[V]^G$, see Corollary 1.

We can use results on modules of covariants to study extensions of rings of invariants $k[V]^G \subset k[V]^H$, where $H < G$. Let $k(G/H)$ be the permutation module on G/H, the collection of the left cosets of H in G. Then the ring of invariants $k[V]^H$ is as a $k[V]^G$-module isomorphic to the module of covariants of type $k(G/H)$, see Lemma 5.

We obtain a generalization of Serre's theorem, see Theorem 4. Suppose that $k[V]^H$ is free as a graded $k[V]^G$-module. Then G is generated by H together with the reflections in G, i.e., $G = HW$. Furthermore, multiplication induces an isomorphism of algebras

$$k[V]^H \otimes_{k[V]^G} k[V]^W \simeq k[V]^{H \cap W}.$$

In particular, in that case $k[V]^W \subset k[V]^{H\cap W}$ is also a free extension and the H-invariants together with the W-invariants generate the ring of $H \cap W$-invariants, i.e.

$$k[V]^{H\cap W} = k[V]^H k[V]^W.$$

Conversely, if $k[V]^W \subset k[V]^{H\cap W}$ is a free extension and the H-invariants together with the W-invariants generate the ring of $H \cap W$-invariants, we have that $k[V]^H$ is free as a graded $k[V]^G$-module, by Proposition 8.

For a further extension of Stanley–Nakajima's results, we first define an analogue of Stanley's polynomial F_M for a general M, introduced by Gutkin [15] over the complex numbers. It is an explicit product of linear forms, defined as follows. For any linear hyperplane U of V, fix a linear form x_U having U as zero-set. As before, we denote the point stabilizer of U by G_U. Put

$$s_U(M) := s_{k[V]^{G_U}}(k[V]^{G_U}(M)).$$

Then we define

$$F_M := \prod_{U \subset V} x_U^{s_U(M)},$$

where the product is over all the linear hyperplanes U of V. It is well-defined up to a non-zero scalar, since $s_U(M) \neq 0$ for only finitely many U's. From equation (3) it follows that the degree of F_M is exactly the s-invariant of $k[V]^G(M)$. Next, we define certain determinants. Fix a basis $v_1, \ldots, v_m$ for M, where $m = \dim_k M$. If $\omega_j := \sum_i f_{ij} \otimes v_i$, $1 \leq j \leq m$, is an m-tuple of elements in $k[V]^G(M)$, write

$$\mathrm{Jac}_M(\omega_1, \ldots, \omega_m) \in k[V]$$

for the determinant of the square $m \times m$ matrix $(f_{ij})_{1 \leq i,j \leq m}$. We call it the *$M$-Jacobian determinant* of the m-tuple of covariants.

Now we are ready to state another generalization of Stanley-Nakajima's freeness criterion, called the *Jacobian criterion of freeness*, see Theorem 3. Let $\omega_1, \ldots, \omega_m \in k[V]^G(M)$ be an m-tuple of homogeneous covariants of type M. They form a free generating set of $k[V]^G(M)$ if and only there is a non-zero constant $c \in k^\times$ such that $\mathrm{Jac}_M(\omega_1, \ldots, \omega_m) = cF_M$ *if and only if* the M-Jacobian determinant of $(\omega_1, \ldots, \omega_m)$ is non-zero and the sum of the degrees of the ω_i's equals the s-invariant of $k[V]^G(M)$.

In this introduction, we described our results on modules of covariants of linear actions on a vector space. But actions on polynomial rings respecting non-standard gradings are also interesting, and likewise the action of G/N on the invariant ring $k[V]^N$ of a normal subgroup $N \lhd G$. That is one of our motivations to work with actions of a finite group on an integrally closed, connected graded algebra without zero-divisors. Many of the results mentioned in this introduction remain true in this more general context. These assumptions imply that all modules of covariants are reflexive. This is a property for modules analogous to the property of normality for an integral domain. A consequence, that we use frequently, is that a morphism between two reflexive modules is an isomorphism if and only if it is an isomorphism at all prime ideals of height one.

1. Preliminaries and the s-invariant

1.1. **Notation.** Fix a base field k of arbitrary characteristic. In this article, all graded algebras $A = \bigoplus_{i \geq o} A_i$ will be assumed to be finitely generated graded algebras over the field k without zero-divisors, and the only elements of degree zero are the scalars, i.e.,

$A_0 = k$. The elements without constant term form the unique maximal homogeneous ideal A_+. Consequently, we have Nakayama's lemma for graded modules at our disposal, and so finitely generated projective graded modules are free. We shall say that A is a *normal graded algebra* if it is a graded algebra integrally closed in its quotient field. A graded A-module $N = \oplus_{i\in\mathbb{Z}} N_i$ will always be assumed to be finitely generated. The *Hilbert series*

$$\mathcal{H}(N;t) := \sum_i \dim_k(N_i)\ t^i$$

is then the Laurent expansion at $t = 0$ of a rational function. For any integer d, the *shifted graded module* $N[d]$ is defined by $N[d]_i := N_{d+i}$. So

$$\mathcal{H}(N[d];t) = \mathcal{H}(N;t)\cdot t^{-d}.$$

Let G be a finite group of graded k-algebra automorphisms on A, and $H < G$ a subgroup. We shall call (the collection of cosets) G/H *modular* if the characteristic of the base field is positive and divides $|G/H| = |G|/|H|$, and *non-modular* otherwise. We shall be interested in the extension of graded algebras $A^G \subset A^H$, where A^G and A^H are the rings of invariants. It will be called a *free (graded) extension* if A^H is a free graded module over A^G; a *graded Gorenstein extension* if it is a free graded extension and the fibre algebra $A^H/A^G_+A^H$ is a Gorenstein algebra (of Krull dimension zero); and a *graded complete intersection extension* if it is a free graded extension and the fibre algebra is a graded complete intersection algebra. We refer to [9] for equivalent definitions and basic properties. For example, if $A^G \subset A^H$ is a free extension, then one of the two invariant rings is Cohen-Macaulay *if and only if* the other is. If it is a graded Gorenstein (resp. graded complete intersection) extension, then one of the two invariant rings is Gorenstein (resp. a complete intersection) *if and only if* the other is. Also several numerical invariants are shared, see Avramov [2].

In most of the following definitions, we no longer need to assume that the k-algebra A is graded. The *inertia subgroup* $G_i(\mathfrak{P})$ of a prime ideal $\mathfrak{P} \subset A$ is defined to be

$$G_i(\mathfrak{P}) := \{\sigma \in G; \forall a \in A:\ \ \sigma(a) - a \in \mathfrak{P}\};$$

it is a normal subgroup of the *decomposition group*

$$G_d(\mathfrak{P}) := \{\sigma \in G; \sigma(\mathfrak{P}) = \mathfrak{P}\}.$$

An element $\sigma \in G$ will be called a *reflection on A* if it is contained in the inertia subgroup of some prime ideal of height one. The subgroup generated by all reflections on A is denoted by W. It is a normal subgroup of G. We shall say that W is the *reflection subgroup* of G acting on A.

The group algebra of G over k is denoted by kG. Let M be a finite dimensional kG-module. We call

$$A^G(M) := (A \otimes_k M)^G$$

the *module of covariants of type* M. It is a finitely generated graded A^G-module isomorphic to $\mathrm{Hom}_{kG}(M^*, A)$, where $M^* = \mathrm{Hom}_k(M, k)$ is the kG-module dual to M. If $\lambda : G \to k^\times$ is a linear character, i.e., a group homomorphism, then

$$A^G_\lambda := \{a \in A; \forall \sigma \in G : \ \sigma(a) = \lambda(\sigma)a\}$$

is called the *module of semi-invariants of type* λ. If k_λ is the one dimensional kG-module on which G acts by the linear character λ and k_λ^* its dual, then

$$A^G(k_\lambda^*) \simeq A^G_\lambda.$$

We denote the collection of all group homomorphisms $\lambda : G \to k^\times$ by $X(G)$; of course, it depends on the base field k. We denote the intersection of the kernels of all $\lambda \in X(G)$ by G^1. It contains the derived group G'; and $G^1 = G'$ if k contains sufficiently many roots of unity. The collection of semi-invariants for all types spans the subalgebra of A^{G^1}, and we have a finite direct sum of A^G-modules

$$A^{G^1} = \bigoplus_{\lambda \in X(G)} A^G_\lambda.$$

Let G be a finite group of linear automorphisms of the vector space V over k. The coordinate ring $k[V]$ is the polynomial algebra $k[x_1, \ldots, x_n]$ on a basis of V^*. This is a graded algebra in the standard way, with G as finite group of automorphisms. It is our standard example of graded algebra. Let $\mathfrak{P} \subset k[V]$ be a prime ideal of height one with non-trivial inertia subgroup. Then $\mathfrak{P} = (f)$ for some irreducible polynomial $f \in k[x_1, \ldots, x_n]$, and there is a non-trivial $\sigma \in G$ such that for all i we have $\sigma(x_i) - x_i \in (f)$ and $\sigma(x_i) \neq x_i$ for at least one i. It follows that $\mathfrak{P}$ is generated by a linear form, say x. Then the decomposition subgroup $G_d(\mathfrak{P})$ is the subgroup of G of elements stabilizing the zero set of x, and the inertia subgroup $G_i(\mathfrak{P})$ is the subgroup of G fixing all elements of the zero set of x. So an element $\sigma \in G$ is a reflection on $k[V]$ *if and only if* σ fixes point-wise some hyperplane of V. The free module $k[V] \otimes_k \wedge^i V^*$ can be identified with the $k[V]$-module of polynomial differential i-forms on V, and $k[V] \otimes_k V$ with the module of polynomial vector fields on V. So $k[V]^G(\wedge^i V^*)$ can be identified with the $k[V]^G$-module of G-invariant polynomial i-forms, and $k[V]^G(V)$ with the module of G-invariant polynomial vector fields on V.

1.2. **The s-invariant.** Let B be a graded algebra of Krull dimension n, and N a graded B-module. Then the numerical invariants $\deg(N)$ and $\psi(N)$ are defined by the Laurent expansion of the Hilbert series of N at $t = 1$:

$$\mathcal{H}(N; t) = \frac{\deg(N)}{(1-t)^n} + \frac{\psi(N)}{(1-t)^{n-1}} + O\left(\frac{1}{(1-t)^{n-2}}\right).$$

For some of the basic properties, see Benson [3]. Closely related are the numerical invariants $r_B(N)$, called the *rank*, and $s_B(N)$, called the *s-invariant*, defined by the Laurent expansion

at $t = 1$:

$$\frac{\mathcal{H}(N;t)}{\mathcal{H}(B;t)} = r_B(N) + s_B(N)(t-1) + O((t-1)^2).$$

Here $r_B(N)$ coincides with the usual rank of N as B-module. The s-invariant was introduced into invariant theory by Brion [7] (but with a difference of sign). We refer to that article for some of the basic properties. For example,

$$s_B(N[d]) = s_B(N) - d r_B(N),$$

where $N[d]$ is the shifted graded module. If N is a free graded B-module of rank r with homogeneous generators of degree e_i, then

$$N \simeq \oplus_{i=1}^r B[-e_i];$$

and so we get the useful formula

$$s_B(N) = \sum_{i=1}^r e_i.$$

If $\mathfrak{P} \subset B$ is a homogeneous prime ideal of height at least one, then $r_B(B/\mathfrak{P}) = 0$ and

$$s_B(B/\mathfrak{P}) = -\frac{\psi(B/\mathfrak{P})}{\deg B}.$$

The relationship between these numerical invariants is as follows.

Lemma 1. *Let B be a graded algebra, N a graded B-module and $\rho : B_1 \to B$ a homomorphism of graded algebras.*

(i) We have $\deg N = r_B(N) \deg B$ *and* $\psi(N) = r_B(N)\psi(B) - s_B(N)\deg(B)$.

(ii) We have $r_{B_1}(N) = r_B(N) r_{B_1}(B)$ *and* $s_{B_1}(N) = s_B(N) r_{B_1}(B) + r_B(N) s_{B_1}(B)$.

Proof. This follows easily by comparing the Laurent series expansions at $t = 1$. □

A homomorphism of graded B-modules $\phi : M \to N$ is called *pseudo-injective* if the localization of the kernel vanishes at all homogeneous prime ideals of height at most one. Or, using the numerical invariants, ϕ is pseudo-injective *if and only if*

$$r_B(\operatorname{Ker}\phi) = s_B(\operatorname{Ker}\phi) = 0.$$

The notions *pseudo-surjective* and *pseudo-isomorphism* are defined similarly.

Let M and N be two graded B-modules. Then $\operatorname{Hom}_B(M,N)$ is also a graded B-module. Benson and Crawley-Boevey [3] expressed the ψ-invariant of this module in terms of the numerical invariants of N, M and B. We write their result in terms of the s-invariants.

Proposition 1. *Let B be a normal graded domain and M and N two graded B-modules.*

(i) Then

$$s_B(\operatorname{Hom}_B(M,N)) - s_B(\operatorname{Ext}^1_B(M,N)) = r_B(M)s_B(N) - r_B(N)s_B(M).$$

If M is torsion free, then $s_B(\mathrm{Ext}^1_B(M,N))=0$, and

$$s_B(\mathrm{Hom}_B(M,B))=-s_B(M).$$

(ii) And

$$s_B(M\otimes_B N)-s_B(\mathrm{Tor}^B_1(M,N))=r_B(M)s_B(N)+r_B(N)s_B(M).$$

Proof. (i) This is just the formula of Benson and Crawley-Boevey [3, Theorem 3.3.2] reformulated in terms of the rank and the s-invariant. Part (ii) is proved similarly as the cited formula for (i). □

The following result also follows from Benson and Crawley-Boevey [3]. Recall that W is the reflection subgroup of G acting on A.

Proposition 2. *Let A be a graded normal domain with a finite group G of automorphisms.*

(i) Let K be a subgroup such that $W\leq K\leq G$. Then

$$s_{A^G}(A^K)=0.$$

(ii) Suppose $G_i(\mathfrak{P})\cap G_i(\mathfrak{P}')=1$ for all distinct homogeneous height one prime ideals $\mathfrak{P}$ and $\mathfrak{P}'$ of A. Then

$$\frac{1}{|G|}s_{A^G}(A)=\sum_{\mathfrak{P}}\frac{1}{|G_i(\mathfrak{P})|}s_{A^{G_i(\mathfrak{P})}}(A),$$

where the sum is over all homogeneous prime ideals of height one.

(iii) Suppose A is factorial. Put δ_G for the degree of a generator of the Dedekind different $\mathfrak{D}_{A/A^G}$, and define similarly δ_H for a subgroup $H<G$. Then

$$s_{A^G}(A^H)=\frac{|G|}{2|H|}(\delta_G-\delta_H).$$

Proof. (i) Since the Dedekind different of the extension $A^G\subset A^K$ is trivial, it follows from [3, Theorem 3.12.1] that

$$|G|\psi(A^G)=|K|\psi(A^K).$$

Or in terms of the s-invariant,

$$s_{A^G}(A^K)=s_{A^K}(A^K)=0.$$

(ii) This is a reformulation in terms of the s-invariant of [3, Corollary 3.12.2] using that

$$|G|\psi(A^G)-\psi(A)=\frac{\deg A}{|G|}s_{A^G}(A).$$

(iii) Again from [3, Theorem 3.12.1] it follows that

$$|G|\psi(A^G)-\psi(A)=\frac{\delta_G\deg A}{2}.$$

Combining with the formula above we get

$$s_{A^G}(A)=\frac{|G|\delta_G}{2}.$$

Similarly with G replaced by H. Now applying Lemma 1(ii) gives the result. □

We partially generalize this result to modules of covariants as follows. A proof is given in Section 5.

Theorem 1. *Let A be a graded normal algebra with finite group G of graded k-algebra automorphisms, and M a finite dimensional kG-module.*

(i) Let K be a subgroup such that $W \leq K \leq G$. Then

$$s_{A^G}(A^G(M)) = s_{A^K}(A^K(M)).$$

In particular, if W acts trivially on M, then $s_{A^G}(A^G(M)) = 0$.

(ii) Suppose $G_i(\mathfrak{P}) \cap G_i(\mathfrak{P}') = 1$ for all distinct homogeneous height one prime ideals $\mathfrak{P}$ and $\mathfrak{P}'$ of A. Then

$$s_{A^G}(A^G(M)) = \sum_{\mathfrak{P}} s_{A^{G_i(\mathfrak{P})}}(A^{G_i(\mathfrak{P})}(M)),$$

where the sum is over the homogeneous height one prime ideals of A.

Proof. See Corollary 2 in the last section. □

In the linear case the s-invariant of a module of covariants is always a non-negative integer.

Proposition 3. *Let G be a finite group of linear automorphisms of the vector space V and M a finite dimensional kG-module. Then $s_{k[V]^G}(k[V]^G(M))$ is a non-negative integer and $s_{k[V]^G}(k[V]^G(M)) = 0$* if and only *if the reflection subgroup W acts trivially on M.*

Proof. We start with a special case. If $k[V]^G(M)$ is free over $k[V]^G$ with homogeneous generators $\omega_1, \dots, \omega_m$, then the s-invariant of $k[V]^G(M)$ is $\sum_i \deg(\omega_i)$. Since $k[V]^G(M)$ has no elements of negative degree it follows that the s-invariant is a non-negative integer. If the s-invariant is 0, then each generator is of degree 0 and hence of the form

$$\omega_i = 1 \otimes u_i \in (k[V] \otimes M)^G.$$

Here each $u_i \in M^G$, and $u_1, \dots, u_m$ form a basis of M. It follows that G acts trivially on M.

Now we no longer assume freeness. The conditions of Theorem 1(ii) are satisfied for a linear action. By Proposition 6(i) or Hartmann–Shepler [17], for every homogeneous prime ideal $\mathfrak{P} \subset k[V]$ of height one the module of covariants $k[V]^{G_i(\mathfrak{P})}(M)$ is free and so by the special case above, $s_{k[V]^{G_i(\mathfrak{P})}}(k[V]^{G_i(\mathfrak{P})}(M)) \geq 0$ and $s_{k[V]^{G_i(\mathfrak{P})}}(k[V]^{G_i(\mathfrak{P})}(M)) = 0$ *if and only if* $G_i(\mathfrak{P})$ acts trivially on M. Since the $G_i(\mathfrak{P})$'s generate W, applying Theorem 1 gives the result. □

A short exact sequence of kG-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

gives rise to a left exact sequence of modules of covariants

$$0 \to A^G(M_1) \to A^G(M_2) \to A^G(M_3).$$

Lemma 2. *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of finite dimensional kG-modules. Put K for the kernel of the action of G on M_2. If the relative trace ideal $\mathrm{Tr}^G_K(A^K) \subseteq A^G$ has height bigger than 1, then

$$s_{A^G}(A^G(M_2)) = s_{A^G}(A^G(M_1)) + s_{A^G}(A^G(M_3)).$$

Here $\mathrm{Tr}^G_K(a) = \sum_{\sigma K \in G/K} \sigma(a)$, for $a \in A^K$.

Proof. The normal subgroup K also acts trivially on M_1 and M_3. So

$$(A \otimes_k M_i)^K \simeq A^K \otimes M_i$$

and we get a short exact sequence

$$0 \to A^K \otimes_k M_1 \to A^K \otimes_k M_2 \to A^K \otimes_k M_3 \to 0$$

of G/K-modules. From the theory of group cohomology we get a long exact sequence of graded $A^G = (A^K)^{G/K}$-modules

$$0 \to A^G(M_1) \to A^G(M_2) \to A^G(M_3) \to H^1(G/K, A^K \otimes M_1) \to H^1(G/K, A^K \otimes M_2) \ldots.$$

It is known that all $H^i(G/K, A^K \otimes M_j)$, $i \geq 1$, are annihilated by the ideal $\mathrm{Tr}^G_K(A^K) \subseteq A^G$, e.g. [12] or [21, Lemma 1.3]. So the cokernel of

$$A^G(M_2) \to A^G(M_3)$$

is annhilated by $\mathrm{Tr}^G_K(A^K)$ as well. Our hypothesis implies therefore that the support of the cokernel of

$$A^G(M_2) \to A^G(M_3)$$

is of codimension ≥ 2, and hence is pseudo-zero with vanishing s-invariant. $\square$

Example 1. Let $k = \mathbb{F}_2$, $V = \mathbb{F}_2^4$, and G the abelian group generated by the three reflections

$$\sigma_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Let x_1, x_2, x_3, x_4 be the basis of V^* dual to the standard basis of V. Then $k[V]$ is the polynomial ring $\mathbb{F}_2[x_1, x_2, x_3, x_4]$. The only homogeneous prime ideals of height one in $k[V]$ with non-trivial inertia subgroups are the principal ideals $\mathfrak{P}_1 = (x_1)$, $\mathfrak{P}_2 = (x_2)$ and $\mathfrak{P}_3 = (x_1 + x_2)$. The inertia subgroup H_i of $\mathfrak{P}_i$ is $\{1, \sigma_i\}$, for $i = 1$, 2 or 3. Let

$$M_1 := \mathbb{F}_2 x_1 + \mathbb{F}_2 x_2 + \mathbb{F}_2 x_3.$$

Then M_1 is a kG-submodule of $M_2 := V^*$, and the quotient module $M_3 = M_2/M_1$ is the trivial kG-module.

Free generators of $(k[V]\otimes M_1)^{H_i}$ are $1\otimes x_1$, $1\otimes x_2$, together with $1\otimes x_3$ if $i=1$, or with $x_2\otimes x_3+x_3\otimes x_2$ if $i=2$, or with $(x_1+x_2)\otimes x_3+x_3\otimes(x_1+x_2)$ if $i=3$. So

$$s_{k[V]^G}(k[V]^G(M_1))=0+1+1=2.$$

Free generators of $(k[V]\otimes M_2)^{H_i}$ are $1\otimes x_1$, $1\otimes x_2$, together with $1\otimes x_3$ and $x_1\otimes x_4+x_4\otimes x_1$ if $i=1$, or with $x_2\otimes x_3+x_3\otimes x_2$ and $1\otimes x_4$ if $i=2$, or with $(x_1+x_2)\otimes x_3+x_3\otimes(x_1+x_2)$ and $1\otimes(x_3+x_4)$ if $i=3$. So

$$s_{k[V]^G}(k[V]^G(M_2))=1+1+1=3.$$

Since M_3 is the trivial kG-module, we have

$$s_{k[V]^G}(k[V]^G(M_3))=0.$$

Indeed, in this case

$$s_{A^G}(A^G(M_2))\neq s_{A^G}(A^G(M_1))+s_{A^G}(A^G(M_3)).$$

So the ideal $\mathrm{Tr}^G(k[V])\subset k[V]^G$ must be of height one.

Example 2. Let $k=\mathbb{F}_p$ be the prime field of order p, V an n dimensional vector space over $\mathbb{F}_p$, and G a group of $\mathbb{F}_p$-linear automorphisms of V. Let $U\subset V$ be a linear hyperplane defined by the linear form x_U, and let G_U be its point stabilizer. It has a semi-direct product decomposition

$$G_U=(\mathbb{Z}/p\mathbb{Z})^{a_U}\rtimes\mathbb{Z}/h_U\mathbb{Z},$$

where $a_U\leq n-1$ and h_U divides $p-1$. Let ζ be a primitive h_U-th root of unity in $\mathbb{F}_p$. We can choose generators τ and $\sigma_1,\ldots,\sigma_{a_U}$ of G_U and a basis $x_1=x_U,x_2,\ldots,x_n$ of coordinate functions, such that

$$\tau x_1=\zeta x_1,\ \sigma_i x_{i+1}=x_{i+1}+x_1,\ 1\leq i\leq a_U,$$

and all other actions are trivial. Then a free basis for $(\mathbb{F}_p[V]\otimes V^*)^{G_U}$ is

$$x_1^{h_U-1}\otimes x_1,\ x_{i+1}\otimes x_1-x_1\otimes x_{i+1},\ \text{if } 1\leq i\leq a_U,\ \text{and } 1\otimes x_i \text{ if } i>a_U+1.$$

Then

$$s_{k[V]^{G_U}}(k[V]^{G_U}(V^*))=(h_U-1+a_U)$$

and so

$$s_{\mathbb{F}_p[V]^G}(\mathbb{F}_p[V]^G(V^*))=\sum_{U\subset V}(h_U-1+a_U),$$

where the sum is over the linear hyperplanes of V. On the other hand, Benson and Crawley-Boevey [3, Theorem 3.13.2] calculated that

$$s_{\mathbb{F}_p[V]^G}(\mathbb{F}_p[V])=\frac{|G|}{2}\delta,$$

where

$$\delta = \sum_{U \subset V} (h_U - 1 + (p-1)a_U)$$

is the degree of a generator of the Dedekind different $\mathfrak{D}_{k[V]/k[V]^G}$. We conclude that if either $p = 2$ or G is transvection free, then

$$s_{\mathbb{F}_p[V]^G}(\mathbb{F}_p[V]^G(V^*)) = \delta.$$

2. Reflexive modules of covariants

Let A be a normal graded algebra, and A^G the invariant ring of a finite group action. A well-known fundamental fact is that A^G is also a normal graded algebra, see [3]. Less well-known is that all modules of covariants $A^G(M) = (A \otimes_k M)^G$ inherit an analogous good property, namely that they are reflexive finitely generated graded modules, see e.g. Brion [7]. In this section we shall reprove this property, and use the fundamental property that a quasi-isomorphism between reflexive modules is already an isomorphism.

We recall that a finitely generated module M over a ring R is called *reflexive* if the natural map $M \to M^{**}$ is an isomorphism, where $M^* = \mathrm{Hom}_R(M, R)$ is the dual module. For example, free modules are reflexive and reflexive modules are torsion free. For more information, see [5, VII §4] or [3, §3.4]. That reflexivity is the module analogue of normality is shown clearly in the next lemma.

Lemma 3. *Let B be a normal graded algebra with quotient field K and N a finitely generated torsion free graded module. Then the following are equivalent.*

(i) N is reflexive;

(ii) $N = \cap_{\mathfrak{p}} N_{\mathfrak{p}}$, where the intersection is inside the vector space $N \otimes_B K$ running over all prime ideals $\mathfrak{p} \subset B$ of height one ;

(iii) Every regular sequence of length two on B is also a regular sequence of length two on N.

Proof. See Samuel [27, Proposition 1]. In the proof it is not needed that B is graded. □

In the applications, we shall mainly use the following properties of reflexive modules.

Lemma 4. *Let B be a normal graded algebra and M and N two reflexive graded B-modules.*

(i) If $\phi : M \to N$ is a pseudo-isomorphism, then it is an isomorphism.

(ii) The B-module $\mathrm{Hom}_B(M, N)$ is also a reflexive graded module.

(iii) If $B \subset A$ is a finite free graded extension, then $A \otimes_B M$ is a reflexive A-module.

(iv) Let $B \subset A$ be a finite extension of normal graded algebras whose extension of quotient fields is separable. A graded A-module is reflexive as A-module if and only if *it is reflexive considered as B-module by restriction.*

Proof. (i) Let K be the quotient field of B. Since the ranks of the kernel and the cokernel are zero, it follows that

$$K \otimes_B M \simeq K \otimes_B N.$$

Identify both K-vector spaces and write $V = K \otimes_B M$. Since M is reflexive,

$$M = \cap_{\mathfrak{P}} M_{\mathfrak{P}},$$

where the intersection is taken inside V and $\mathfrak{P}$ runs over all height one prime ideals, cf. [5, VII §4.2 Théorème 2]. The same for N. Since ϕ is a pseudo-isomorphism we have, for each height one prime ideal $\mathfrak{P}$ in B, that $M_{\mathfrak{P}} = N_{\mathfrak{P}} \subset V$. Hence $M = N$ and ϕ is an isomorphism.

For (ii), use [3, Lemma 3.4.1(v)] and for (iii), use [5, VII §4.2 Proposition 8].

(iv) Let M be a graded A-module. For any height one prime ideal $\mathfrak{p} \subset B$ let $\mathfrak{P}_1, \dots, \mathfrak{P}_t$ be the prime ideals in A lying over $\mathfrak{p}$. Then all $\mathfrak{P}_i$'s are of height one. It suffices to prove that

$$M_{\mathfrak{p}} = \cap_i M_{\mathfrak{P}_i}.$$

Let $\frac{m}{s}$ be in the intersection, with $m \in M$ and $s \in A$, $s \neq 0$. For every i there exists an $m_i \in M$ and $s_i \in A \backslash \mathfrak{P}_i$ such that $\frac{m}{s} = \frac{m_i}{s_i}$. Let I be the ideal in A generated by $s_1, s_2, \dots$. Then I is not contained in the union of the $\mathfrak{P}_i$'s, by the prime avoidance lemma. So there is a $u = \sum_i x_i s_i \in I$ that is not in the union of the $\mathfrak{P}_i$'s, hence in none of them. Then

$$um = \sum_i x_i s_i m = s \sum_i x_i m_i$$

and so

$$\frac{m}{s} = \frac{\sum_i x_i m_i}{u}.$$

Hence, we can assume that s is not in any of the $\mathfrak{P}_i$'s, nor that any of the conjugates of s is in any of the $\mathfrak{P}_i$'s. Using the norm of s, we can assume that s is in B and not in any of the $\mathfrak{P}_i$'s, hence not in $\mathfrak{p}$. So

$$M_{\mathfrak{p}} = \cap_i M_{\mathfrak{P}_i}.$$

□

2.1. **Modules of covariants are reflexive.** In this subsection, we shall reprove that all modules of covariants are finitely generated reflexive modules over the ring of invariants, using Samuel's Lemma 3. For another proof, see Brion [7].

Proposition 4. *Suppose A is a normal graded algebra. Every module of covariants $A^G(M)$ is a reflexive, finitely generated graded A^G-module of rank equal to* $\dim_k M$.

Proof. We first prove that $A^G(M)$ is a finitely generated, torsion free, graded A^G-module of rank $\dim_k M$. Since $A^G(M)$ is a submodule of the torsion free A^G-module $A \otimes_k M$, it is torsion free. Let L be the quotient field of A. Then $K := L^G$ is the quotient field of A^G, and

$$A^G(M) \otimes_{A^G} K \simeq (L \otimes_k M)^G.$$

By the normal basis theorem of Galois theory, there exists a $z \in L$ such that $\{\sigma(z); \sigma \in G\}$ is a K-basis of L. If $v_1, \ldots, v_m$ is a k-basis for M, then it is easily shown that

$$\{\sum_{\sigma \in G} (\sigma(z) \otimes \sigma(v_i)),\ 1 \leq i \leq m\}$$

is a K-basis for $(L \otimes_k M)^G$. We conclude that $A^G(M)$ has rank $\dim_k M$ over A^G.

Let

$$S = \oplus_{i \geq 0}(A \otimes_k S^i M) = A \otimes_k \oplus_{i \geq 0} S^i M$$

be the symmetric algebra on the free A-module $A \otimes_k M$. It is a finitely generated algebra, therefore its invariant ring

$$S^G = \oplus_{i \geq 0} A^G(S^i M)$$

is a finitely generated algebra over its subalgebra A^G. The homogeneous A^G-algebra generators of S^G that are contained in $A^G(M)$ are also homogeneous A^G-module generators of $A^G(M)$. So $A^G(M)$ is a finitely generated A^G-module.

By Lemma 3 a finitely generated, torsion free graded module is reflexive if every regular sequence u, v on the ring is also a regular sequence on the module. Let u, v be a regular sequence on A^G. Since $A^G(M)$ is torsion free, u acts regularly on $A^G(M)$. Let $\omega_1, \omega_2 \in A^G(M)$ such that $v\omega_1 = u\omega_2$. Since A is a reflexive A^G-module, cf. [5, VII §4.8 Corollaire], u, v is a regular sequence on $A \otimes_k M$. So, there is an $\omega \in A \otimes_k M$ such that $\omega_1 = u\omega$. If ω is not in $A^G(M)$, then there exists a $\sigma \in G$ such that $\omega \neq \sigma(\omega)$. But $\omega_1 = \sigma(\omega_1)$ and $u = \sigma(u)$, so $u(\omega - \sigma(\omega)) = 0$. Since u acts regularly on $A \otimes_k M$, we get $\omega = \sigma(\omega)$, which is a contradiction. So $\omega \in A^G(M)$ and u, v is a regular sequence on $A^G(M)$. We conclude that $A^G(M)$ is reflexive. □

2.2. **The Cohen-Macaulay property in the non-modular situation.** Since modules of covariants are reflexive, every regular sequence of length two on A^G is also a regular sequence on any non-zero module of covariants. One might ask whether the same holds for longer regular sequences. In the non-modular situation it is known that if A is Cohen-Macaulay then A^G is also Cohen-Macaulay, by Hochster-Eagon [18]. More generally, in that context, all modules of covariants are Cohen-Macaulay. In the non-modular situation it is also known that if A is free over A^G, then all modules of covariants are free. Both results are no longer true in the modular situation, we give counter examples later on.

Proposition 5. *Let A be a graded algebra without zero-divisors, G a finite group of automorphisms on A, $H < G$ a subgroup, and M a finite dimensional kG-module. Suppose that G/H is non-modular.*

(i) Suppose $A^H(M)$ is a Cohen-Macaulay A^H-module. Then $A^G(M)$ is a Cohen-Macaulay A^G-module. In particular, if A^H is Cohen-Macaulay, then A^G is also Cohen-Macaulay. If, in addition, H acts trivially on M, then $A^G(M)$ is a Cohen-Macaulay A^G-module.

(ii) Suppose $A^G \subset A^H$ is a free graded extension, and $A^H(M)$ free over A^H. Then $A^G(M)$ is free over A^G. In particular, if H acts trivially on M, then $A^G(M)$ is free.

Proof. Since G/H is non-modular, the operator

$$\frac{1}{|G/H|}\operatorname{Tr}_H^G : (A \otimes_k M)^H \to (A \otimes_k M)^G : \omega \mapsto \frac{1}{|G/H|}\sum_{gH \in G/H} g\omega$$

is A^G-linear and the identity when restricted to the submodule $(A \otimes_k M)^G$. So $A^G(M)$ is an A^G-direct summand of $A^H(M)$.

(i) Suppose $A^H(M)$ is a Cohen-Macaulay A^H-module. Let $f_1, \ldots, f_n$ be a homogeneous system of parameters for A^G. We have to show that $f_1, \ldots, f_n$ is a regular sequence on $A^G(M)$. Let $R = k[f_1, \ldots, f_n]$ be the polynomial algebra it generates. Then $A^H(M)$ is also a Cohen-Macaulay R-module. By the Auslander-Buchsbaum equation, it follows that $A^H(M)$ is a projective graded R-module, hence free, by Nakayama's lemma for connected graded algebras, see [13]. Since a direct summand of a free graded R-module is also a free graded R-module, it follows that $A^G(M)$ is a free graded R-module. In particular, $f_1, \ldots, f_n$ forms a regular sequence on $A^G(M)$, and so $A^G(M)$ is a Cohen-Macaulay A^G-module.

(ii) If A^H is free over A^G, then the direct summand $A^G(M)$ of the free graded A^G-module $A^H(M)$ is also free. □

2.3. **Linear actions with large fixed point spaces.** In the linear situation, we can give some additional general results. If G acts linearly on V with fixed points space V^G of codimension one, then $k[V]^G$ is a polynomial ring and all its modules of covariants are free. A similar result holds if $\operatorname{codim}_V V^G = 2$ and $k[V]^G$ is a polynomial algebra. The algebraic-geometric proof of the result depends in an essential way on the linearity of the action and is based on an idea going back to Nakajima [24].

Proposition 6. *Suppose G acts linearly on V, and let M be a finite dimensional kG-module.*

(i) Suppose $\operatorname{codim}_V V^G = 1$. Then $k[V]^G$ is a polynomial algebra and $(k[V] \otimes M)^G$ is free as a graded module over $k[V]^G$.

(ii) Suppose $\operatorname{codim}_V V^G = 2$. Then $(k[V] \otimes M)^G$ is a Cohen-Macaulay graded module over $k[V]^G$. In particular, $k[V]^G$ is a Cohen-Macaulay algebra. Furthermore, if $k[V]^G$ is a polynomial algebra, then every module of covariants is free.

Proof. Let R be a graded algebra with maximal ideal R_+ and consider $k = R/R_+$ as R-module. Then a finitely generated graded R-module M is free over R *if and only if* $\operatorname{Tor}_i^R(M, k) = 0$, for $i \geq 1$. Let $k \subset K$ be a field extension, put $R_K = R \otimes_k K$ and $M_K = M \otimes_k K$. Then $\operatorname{Tor}_i^R(M, k) \otimes_k K \simeq \operatorname{Tor}_i^{R_K}(M_K, K)$. So M is free over R *if and only if* M_K is free over R_K. Let $f_1, \ldots, f_n$ be a homogeneous system of parameters of $k[V]^G$. Then it is also a homogeneous system of parameters for $K[V]^G$. Put $R = k[f_1, \ldots, f_n]$. Then $k[V]^G(M)$ is a Cohen-Macaulay graded $k[V]^G$-module *if and only if* it is free over R. Finally

$k[V]^G$ is polynomial *if and only if* $k[V]$ is a free $k[V]^G$-module. From these remarks it follows that we can, without loss of generality, suppose that k is algebraically closed.

The orbit space V/G is an affine algebraic variety with coordinate ring $k[V]^G$. The morphism of algebraic varieties $V \to V/G$ associating v to its orbit Gv corresponds to the inclusion $k[V]^G \subset k[V]$. The linear algebraic group $U := V^G$ acts on V by translation, commuting with the G-action, so U also acts on the orbit space V/G with orbits of dimension $\dim V^G$.

(i) The singular locus of V/G is closed, see [13, Cor. 16.20], and U-stable. So, if it is non-empty, its dimension is at least $\dim V^G = \dim V - 1$. But, since V/G is normal, its singular locus is of codimension at least two, see [13, Th. 11.5]. This gives a contradiction. So V/G is non-singular. Therefore $k[V]^G$ is a regular graded algebra, hence a polynomial algebra. Analogously, the non-free locus of $(k[V] \otimes M)^G$ is a closed set, defined by a suitable Fitting ideal of a finite presentation, see [13, Prop. 20.8] or [22, Theorem 4.10], and stable under the action of U. So, if it is not empty, its codimension is one. But, since it is torsion free and every finitely generated torsion free module over a discrete valuation ring is free, it follows that $(k[V] \otimes M)^G$ is free in codimension one. A contradiction. So freeness follows.

(ii) Under the hypothesis of (ii), the algebraic group U acts on V/G with orbits of codimension two. Suppose $(k[V] \otimes M)^G$ is not Cohen-Macaulay. Then its non-Cohen-Macaulay locus in V/G is non-empty, U-stable, closed (see [14, Cor. 6.11.3]), and contains $\pi(0)$; so contains the whole of $\pi(V^G)$. Let $\mathfrak{P}$ be the linear ideal defining V^G; it is a prime ideal of height 2. We conclude that $(k[V] \otimes M)^G$ is not Cohen-Macaulay at $\mathfrak{p} := \mathfrak{P} \cap k[V]^G$.

Write B for the localisation of $k[V]^G$ at $\mathfrak{p}$ and N for the localisation of $k[V]^G(M)$ at $\mathfrak{p}$. Then B is a normal local ring of dimension two, hence Cohen-Macaulay and N is a reflexive B-module. And so every regular sequence (of length two) is also a regular sequence on N, by Lemma 3 (this lemma remains true in the case where B is only a noetherian integrally closed domain). We conclude that N is Cohen-Macaulay. But this is a contradiction; so $(k[V] \otimes M)^G$ is Cohen-Macaulay. We conclude by remarking that every graded Cohen-Macaulay module for a polynomial algebra is free, by Auslander-Buchsbaum's formula, cf. [13, Thm. 19.9]. $\square$

Remark. The freeness in (i) was proved differently by Hartmann and Shepler [17].

A similar proof shows that the freeness (or Cohen-Macaulay) property of modules of covariants for G descends to modules of covariants of point-stabilizers.

Proposition 7. *Suppose G acts linearly on V. Let $U \subset V$ be a linear subspace with point-stabilizer G_U, and let M be a finitely generated kG-module.*

(i) If $k[V]^G(M)$ is free over $k[V]^G$, then $k[V]^{G_U}(M)$ is free over $k[V]^{G_U}$.

(ii) If $k[V]^G(M)$ is Cohen-Macaulay over $k[V]^G$, then $k[V]^{G_U}(M)$ is Cohen-Macaulay over $k[V]^{G_U}$.

Proof. As in the proof of Proposition 6, we can assume that k is algebraically closed. The linear algebraic group U acts on V by translations, commuting with the G_U-action. Let $\mathfrak{P}$ be the prime ideal generated by the linear forms vanishing on U, and put $\mathfrak{p} = \mathfrak{P} \cap k[V]^{G_U}$. Let $v \in U$ such that $G_v = G_U$ (we can find such a v since k is algebraically closed), and let $\mathfrak{M}_v \supset \mathfrak{P}$ be the corresponding maximal ideal. Then G_U coincides with the decomposition subgroup of $\mathfrak{M}_v$. Put $\mathfrak{m}_v := \mathfrak{M}_v \cap k[V]^{G_U}$; it is a maximal ideal containing $\mathfrak{p}$. There exists a translation in U that moves the maximal graded ideal of $k[V]^{G_U}$ onto $\mathfrak{m}_v$.

Suppose $k[V]^{G_U}(M)$ is not free (or Cohen-Macaulay), then it is not free (or Cohen-Macaulay) at the maximal graded ideal and (using the commuting translation action) therefore not free (or Cohen-Macaulay) at the maximal ideal $\mathfrak{m}_v$ either. This gives a contradiction with Lemma 6 in the last section. □

3. Generalizations of results of Stanley and Nakajima

3.1. **Free modules of covariants.** Stanley [33] and Nakajima [23] already extensively studied modules of semi-invariants. One of their fundamental results can be described as follows. Let $\lambda : G \to k^\times$ be a linear character with module of semi-invariants A^G_λ. Then if A^G_λ is free, A^W_λ is also free and both modules of semi-invariants share the generator. For modules of covariants, this is generalized as follows.

Theorem 2. *Let A be a normal graded algebra, G a finite group of automorphisms of A, and M a finite dimensional kG-module. Let K be a subgroup such that $W \leq K \leq G$. Suppose that $A^G(M)$ is free over A^G. Then multiplication induces an isomorphism of graded A^K-modules*

$$\mu : A^K \otimes_{A^G} A^G(M) \simeq A^K(M).$$

In particular, $A^K(M)$ is free over A^K, and $A^G(M)$ and $A^K(M)$ share bases.

Proof. By assumption, $A^K \otimes_{A^G} A^G(M)$ is free over A^K of rank $\dim_k M$. So both sides are reflexive of the same rank. And since

$$s_{A^K}(A^K \otimes_{A^G} A^G(M)) = s_{A^G}(A^G(M)) = s_{A^K}(A^K(M)),$$

by Theorem 1, both sides also have the same s-invariant. Let L be the quotient field of A. To prove that μ generically is an isomorphism, we must show that the L^K-linear map

$$L^K \otimes_{L^G} (L \otimes_k M)^G \to (L \otimes_k M)^K \tag{4}$$

is an isomorphism of L^K-vector spaces of dimension $m = \dim_k M$. It would follow that μ is a pseudo-isomorphism, and then, by Lemma 4(i), that μ is an isomorphism. To show that (4) is an isomorphism, it suffices to show injectivity, or that any L^G-basis of $(L \otimes_k M)^G$ is also an L^K-basis of $(L \otimes_k M)^K$. Let $\omega_1, \ldots, \omega_m$ be an L^G-basis of $(L \otimes_k M)^G$. Suppose we

have an L^K-linear relation $\sum_i u_i\omega_i = 0$, where $u_i \in L^K$. Since the ω_i's are G-invariant, it follows for all $v \in L^K$ that

$$\sum_i \mathrm{Tr}_K^G(vu_i)\omega_i = 0, \quad \text{where } \mathrm{Tr}_K^G := \sum_{\sigma K \in G/K} \sigma : L^K \to L^G.$$

Since the ω_i are independent over L^G, it follows that for all $v \in L^K$ and i we have

$$\mathrm{Tr}_K^G(vu_i) = 0.$$

But since the field extension $L^G \subset L^K$ is separable, the map $\mathrm{Tr}_K^G : L^K \to L^G$ is non-zero, by [20, VI Theorem 5.2]. So each u_i is zero, and the $\omega_1, \ldots, \omega_m$ are also independent over L^K, hence forms a basis. This finishes the proof. □

3.2. **Jacobian criterion for freeness.** Let G act linearly on the vector space V, and let $\lambda : G \to k^\times$ be a linear character. Stanley [33] and Nakajima [23] show that $k[V]_\lambda^W$ is always free of rank one over $k[V]^W$, and they construct a generator f_λ of $k[V]_\lambda^W$. It is a product of linear forms; let e_λ be its degree. They prove that $k[V]_\lambda^G$ is free over $k[V]^G$ *if and only if* $k[V]_\lambda^G$ contains a non-zero element of degree e_λ; in that case f_μ is the generator.

This freeness criterion was generalized, in some sense, to modules of covariants in characteristic zero by Gutkin [15], [25]. Let M be an m-dimensional kG-module with a fixed basis $v_1, \ldots, v_m$. For m homogeneous elements

$$\omega_j = \sum_{i=1}^m a_{ij} \otimes v_i \in k[V] \otimes_k M, \ 1 \le j \le m,$$

define the *Jacobian determinant* by

$$\mathrm{Jac}_M(\omega_1, \ldots, \omega_m) := \det{(a_{ij})}_{1 \le i,j \le m} \in k[V].$$

We formulate the Jacobian criterion of freeness. $k[V]^G(M)$ is free over $k[V]^G$ *if and only if* there is an m-tuple $\omega_1, \ldots, \omega_m$ of homogeneous elements in $k[V]^G(M)$ whose Jacobian determinant $\mathrm{Jac}_M(\omega_1, \ldots, \omega_m)$ is non-zero and of degree e_M, where e_M is the s-invariant $s_{k[V]^G}(k[V]^G(M))$.

Next, we associate to M a certain product of linear forms F_M. Let $U \subset V$ be a linear hyperplane defined by the linear form x_U and let G_U be its point-stabilizer. The module of covariants $k[V]^{G_U}(M)$ is free, by Proposition 6, with homogeneous generators $w_1, \ldots, w_m$, say. Put $e_U(M)$ for the sum of their degrees. We claim that $\mathrm{Jac}_M(w_1, \ldots, w_m)$ is equal to $x_U^{e_U(M)}$, up to a non-zero scalar in $k^\times$. There are only finitely many U's such that $G_U \ne 1$, so the following product of linear forms is well-defined

$$F_M := \prod_{U \subset V,\ \mathrm{codim}_V U = 1} x_U^{e_U(M)}.$$

Its degree turns out to be $e_M = \sum_U e_U(M)$.

Write $\lambda : G \to k^\times$ for the linear character associated to the kG-module $(\wedge^m M)^*$. If $\omega_1, \ldots, \omega_m$ is an m-tuple of homogeneous elements of $k[V]^G(M)$, then the Jacobian determinant $\mathrm{Jac}_M(\omega_1, \ldots, \omega_m)$ is a λ-semi-invariant. Put J_M^G for the $k[V]^G$-submodule of $k[V]_\lambda^G$ spanned by all such Jacobian determinants. We'll show that any element of J_M^G is divisible by F_M.

The following generalizes a result due to Gutkin in characteristic zero [15], [25].

Theorem 3. *Let G act linearly on V, and let M be a finite dimensional kG-module.*

(i) For any linear subspace $U \subset V$ of codimension one we have

$$J_M^{G_U} = k[V]^{G_U} \cdot x_U^{e_U(M)} \subseteq k[V]_{\lambda_U}^{G_U},$$

where $\lambda_U = \lambda|_{G_U}$ and $e_U(M)$ equals the s-invariant $s_{k[V]^{G_U}}(k[V]^{G_U}(M))$.

(ii) F_M is a greatest common divisor of all the elements in J_M^G and

$$J_M^G \subseteq k[V] \cdot F_M.$$

(iii) [Jacobian criterion] *Let $\omega_1, \ldots, \omega_m$ be an m-tuple of homogeneous elements of $k[V]^G(M)$. The following three statements are equivalent.*

(a) The module of covariants $k[V]^G(M)$ is free over $k[V]^G$ with basis $\omega_1, \ldots, \omega_m$;

(b) There is a non-zero scalar $c \in k^\times$ such that

$$\mathrm{Jac}_M(\omega_1, \ldots, \omega_m) = cF_M;$$

(c) $\sum_{i=1}^m \deg \omega_i = s_{k[V]^G}(k[V]^G(M))$ and $\mathrm{Jac}_M(\omega_1, \ldots, \omega_m) \neq 0$.

Proof. The proof is given in subsection 5.2. $\square$

We get a quick proof of a part of Theorem 2 in the linear situation, together with its converse.

Corollary 1. *Let G act linearly on V, M be a finite dimensional kG-module, and K be a subgroup such that $W \leq K \leq G$.*

Then $k[V]^G(M)$ is free over $k[V]^G$ if and only if $k[V]^K(M)$ is free over $k[V]^K$ and $k[V]^G(M)$ and $k[V]^K(M)$ share generators.

Proof. Suppose $k[V]^G(M)$ is free over $k[V]^G$ with homogeneous basis $\omega_1, \ldots, \omega_m$. Then we get, by the Jacobian criterion, that $\mathrm{Jac}_M(\omega_1, \ldots, \omega_m)$ is non-zero of degree $s_{k[V]^G}(k[V]^G(M))$. These basis elements are also elements of $k[V]^K(M)$ and we have the same s-invariants

$$s_{k[V]^G}(k[V]^G(M)) = s_{k[V]^K}(k[V]^K(M)),$$

by Theorem 1. By the Jacobian criterion, $k[V]^K(M)$ is free with homogeneous basis $\omega_1, \ldots, \omega_m$ and so $k[V]^K(M)$ and $k[V]^G(M)$ share generators.

Conversely, suppose $A^K(M)$ is free over A^K and $A^G(M)$ and $A^K(M)$ share generators. This means that $A^K(M)$ can be generated by G-invariant elements, and hence we can extract

a homogeneous basis $\omega_1, \ldots, \omega_m$ of $A^K(M)$ consisting of G-invariant elements. From the Jacobian criterion it follows that $\mathrm{Jac}_M(\omega_1, \ldots, \omega_m)$ is non-zero of degree

$$s_{k[V]^K}(k[V]^K(M)) = s_{k[V]^G}(k[V]^G(M)).$$

So, by the Jacobian criterion once again, it follows that the $\omega_1, \ldots, \omega_m$ form a basis of the free $k[V]^G$-module $k[V]^G(M)$. □

Example 3. We continue example 1. Let us prove that $k[V]^{H_2}(M_1)$ indeed has basis $1 \otimes x_1$, $1 \otimes x_2$, and $x_2 \otimes x_3 + x_3 \otimes x_2$, like we claimed before. First of all, the Jacobian determinant of this triple is x_2, so $s(k[V]^{H_2}(M_1)) \leq 1$. But $s(k[V]^{H_2}(M_1)) = 0$ is impossible, since H_2 does not act trivially on M_1. So $s(k[V]^{H_2}(M_1)) = 1$, and by the Jacobian criterion the three given covariants form a basis.

Next, we consider the three G-covariants of type M_1: $1 \otimes x_1$, $1 \otimes x_2$, and $x_4(x_1 + x_4) \otimes x_1 + x_3(x_1 + x_3) \otimes x_2 + x_2(x_1 + x_2) \otimes x_3$. Their Jacobian determinant is $x_2(x_1 + x_2)$, hence non-zero of degree $2 = s_{k[V]^G}(k[V]^G(M_1))$. So, by the Jacobian criterion, they form a basis of the free module of covariants $k[V]^G(M_1)$ and $F_{M_1} = x_2(x_1 + x_2)$.

Similarly, $1 \otimes x_1$, $1 \otimes x_2$, $x_4 \otimes x_1 + x_3 \otimes x_2 + x_2 \otimes x_3 + x_1 \otimes x_4$, and $x_4^2 \otimes x_1 + x_3^2 \otimes x_2 + x_2^2 \otimes x_3 + x_1^2 \otimes x_4$ is a basis of the free module of covariants $k[V]^G(M_2)$, with $F_{M_2} = x_1 x_2 (x_1 + x_2)$.

We remark that $k[V]^G$ is not a polynomial ring, but is minimally generated by x_1, x_2, $x_1 x_4 (x_1 + x_4) + x_2 x_3 (x_2 + x_3)$, $x_3^4 + (x_1^2 + x_1 x_2 + x_2^2) x_3^2 + x_1 x_2 (x_1 + x_2) x_3$ and $x_4^4 + (x_1^2 + x_1 x_2 + x_2^2) x_4^2 + x_1 x_2 (x_1 + x_2) x_4$.

Example 4. Let G act linearly on the vector space V of dimension n. Fix a basis $x_1, \ldots, x_n$ of linear forms. For any invariant $f \in k[V]^G$, write df for the covariant

$$df := \sum_i \frac{\partial f}{\partial x_i} \otimes x_i \in (k[V] \otimes_k V^*)^G = k[V]^G(V^*).$$

Let $f_1, f_2, \ldots, f_n$ be an n-tuple of G-invariants, then

$$\mathrm{Jac}_{V^*}(df_1, \ldots, df_n) = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i,j \leq n}.$$

Write $e_{V^*} := s_{k[V]^G}(k[V]^G(V^*))$. Then our Jacobian criterion says that df_1, df_2, $\ldots$, df_n (freely) generate $k[V]^G(V^*)$ if and only if $\sum_i \deg df_i = \sum_i (\deg f_i - 1) = e_{V^*}$ and

$$\mathrm{Jac}_{V^*}(df_1, \ldots, df_n) \neq 0.$$

On the other hand the classical Jacobian criterion for $k[V]^G$ to be polynomial is as follows, see [10, Criterion 2]. $k[V]^G = k[f_1, \ldots, f_n]$ *if and only if* $\sum_i (\deg f_i - 1) = \delta$ and $\det \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i,j \leq n} \neq 0$, where δ is the differential degree.

We conclude that if $\delta = e_{V^*}$, then $k[V]^G = k[f_1, \ldots, f_n]$ (hence is a polynomial algebra) *if and only if* df_1, df_2, $\ldots$, df_n (freely) generate $k[V]^G(V^*)$.

By example 2, if the base field equals $\mathbb{F}_2$, we have the equality $\delta = e_{V^*}$.

We claim that if G has no transvections, $\delta = e_{V^*}$ also holds. Write det for the linear character associated to the kG-module $\wedge^n V$. Let $U \subset V$ be a linear subspace of codimension one, defined by the linear form x_U. The point-stabilizer G_U is cyclic, say with generator σ of order h_U, since there are no transvections. There is an eigenbasis of linear forms $x_1 = x_U$, $x_2, \ldots, x_n$; the eigenvalue of x_1 is a primitive h_U-th root of unity in k, the other x_i's have eigenvalue one. Then

$$k[V]^{G_U} = k[x_1^{h_U}, x_2, \ldots, x_n]$$

and $k[V]^G(V^*)$ is freely generated by $x_1^{h_U-1} \otimes x_1, 1 \otimes x_2, \ldots, 1 \otimes x_n$ with Jacobian determinant $x_1^{h_U-1}$. But $x_1^{h_U-1}$ is also the generator of the module of det-semi-invariants. We conclude that F_{V^*} equals the generator of W-semi-invariants of type det, and so its degree e_{V^*} equals δ.

Remark. It is conjectured that if $k[V]^G$ is polynomial, at least the module of covariants $(k[V] \otimes V^*)^G$ is free, see [17]. If there are no transvections in G, this is true, by the example above, and was known before by work of Knighten [19] or Hartmann [16]. Knighten proved that if there are no transvections, then $k[V]^G(V^*)$ is isomorphic to $\Omega^{**}_{k[V]^G/k}$, the reflexive closure of the module of differentials. If $k[V]^G$ is a polynomial algebra, then the module of differentials $\Omega_{k[V]^G/k}$ is free, and hence isomorphic to the module of covariants $k[V]^G(V^*)$; which is therefore free. The modules of covariants $k[V]^G(\wedge^i(V^*))$ are then also free; the arguments in characteristic zero given by Shepler [31] carry over to the general transvection free case.

4. Free extensions of algebras of invariants

Let $H < G$ be a subgroup and let G act as before on the graded algebra A. It is of interest to study the ring of invariants A^H as a module over its subring A^G. For example, if A^H is free over A^G, both rings have a very close ring theoretic relationship, e.g., one of the two is Cohen-Macaulay *if and only if* the other is. Fortunately, A^H is as an A^G-module isomorphic to a module of covariants, in particular even A itself, so we can apply the theorems we have obtained before. We start by showing somewhat more generally that modules of covariants of subgroups are modules of covariants.

Lemma 5. *Let A be a graded algebra, G a finite group of automorphisms of A, $H < G$ a subgroup, and M a finite dimensional kH-module. There is an isomorphism of A^G-modules:*

$$(A \otimes_k M)^H \simeq (A \otimes_k \operatorname{Ind}_H^G M)^G.$$

In particular, the ring of invariants A^H is as A^G-module isomorphic to a module of covariants

$$A^H \simeq A^G(\operatorname{Ind}_H^G k).$$

Here we take $\operatorname{Ind}_H^G M$ to be $kG \otimes_{kH} M$, and in particular $\operatorname{Ind}_H^G k$ is the permutation kG-module on the left coset space G/H.

Proof. This can be proved using Frobenius reciprocity of representation theory, or directly as follows. Fix left coset representatives $g_1, \ldots, g_s$ of G/H, where $g_1 = 1$, and a basis $v_1, \ldots, v_m$ of M. We will show that the A^G-linear map

$$\psi : A^H(M) \to A^G(\operatorname{Ind}_H^G M) : \sum_j a_j \otimes v_j \mapsto \operatorname{Tr}_H^G(\sum_j a_j \otimes g_1 \otimes v_j),$$

is an isomorphism. Here $\operatorname{Tr}_H^G = \sum_i g_i$, $a_j \in A$ and $g_1 = 1$ is considered as the unit in kG. Let $\omega \in (A \otimes_k \operatorname{Ind}_H^G M)^G$ be any element. It can be uniquely written as $\omega = \sum_{i,j} a_{ij} \otimes g_i \otimes v_j$, where of course $a_{ij} \in A$, and $g_i \in G$ is seen as a basis element of kG over kH. We can write $\omega = \sum_i \omega_i$, where $\omega_i = \sum_j a_{ij} \otimes g_i \otimes v_j$. Since ω is G-invariant, we get $g_i \omega = \omega$, and $g_i \omega_1 = \omega_i$ (since $g_1 = 1$ and by the unicity of the expression). So

$$\omega = \sum_i g_i \omega_1 = \operatorname{Tr}_H^G \omega_1.$$

It also follows that ω_1 is H-invariant, and so $\sum_j a_{1j} \otimes v_j \in A^H(M)$. Therefore, ψ is surjective. We remark that $\omega = 0$ *if and only if* $\omega_1 = 0$, hence injectivity. □

Remark. Suppose G acts linearly on V such that the invariant ring $k[V]^G$ is polynomial. If G is non-modular then all modules of covariants are free. It is natural to ask, whether this remains true when G is modular. The answer is no.

There are many known examples of groups H acting linearly on a vector space V over a finite field $\mathbb{F}_q$ whose ring of invariants $\mathbb{F}_q[V]^H$ is not Cohen-Macaulay, see [12]. Let $G = \operatorname{GL}(V)$ be the full linear group. Then $\mathbb{F}_q[V]^G$ is a polynomial ring (generated by the so-called Dickson-invariants, see [3]). Since $\mathbb{F}_q[V]^H$ is not Cohen-Macaulay it is not free over the polynomial subring $\mathbb{F}_q[V]^G$. But $\mathbb{F}_q[V]^H$ is a special module of covariants for $\mathbb{F}_q[V]^G$. We conclude that some modules of covariants for $\mathbb{F}_q[V]^G$ are not free, even though $\mathbb{F}_q[V]^G$ is polynomial.

4.1. **Generalization of Serre's theorem.** For a linear action of G on a vector space V the invariant algebra $k[V]^G$ is a polynomial algebra *if and only if* the extension $k[V]^G \subset k[V]$ is free. By a theorem of Serre the group is then generated by reflections, cf. [3, p. 85-86]. The converse is true when G is non-modular. We give a generalization of Serre's result.

Theorem 4. *Let A be a normal graded algebra with a finite group G acting on it by graded algebra automorphisms. Let H and K be subgroups of G such that K contains the reflection subgroup W of G acting on A. Suppose $A^G \subset A^H$ is a free graded extension.*

(i) The group G is generated by H and the reflections on A contained in G, i.e., $G = WH$.

(ii) Multiplication induces an isomorphism

$$A^K \otimes_{A^G} A^H \simeq A^{K \cap H}.$$

In particular, $A^K \subset A^{K\cap H}$ is also a free graded extension, and

$$A^K A^H = A^{K \cap H},$$

i.e., the ring of $K \cap H$-invariants is generated by the K-invariants and the H-invariants.

(iii) Furthermore, $A^G \subset A^H$ is a graded Gorenstein extension (or complete intersection extension) if and only if *$A^K \subset A^{K\cap H}$ is a graded Gorenstein extension (or complete intersection extension).*

Proof. (i) By Lemma 5 we can consider A^H as a module of covariants over A^G by

$$A^H \simeq A^G(\operatorname{Ind}_H^G k).$$

Since $W \leq WH \leq G$ and A^H is free over A^G, we can apply Theorem 2 and obtain an isomorphism of A^{WH}-modules

$$\mu : A^{WH} \otimes_{A^G} A^G(\operatorname{Ind}_H^G k) \simeq A^{WH}(\operatorname{Ind}_H^G k)$$

and so $A^{WH}(\operatorname{Ind}_H^G k)$ is a free graded A^{WH}-module. We remark that $\operatorname{Ind}_H^G k$ is a permutation kWH-module containing the permutation submodule $\operatorname{Ind}_H^{WH} k$, which is therefore a kWH-direct summand. It follows that $A^{WH}(\operatorname{Ind}_H^{WH} k)$ is a direct summand of $A^{WH}(\operatorname{Ind}_H^G k)$ as graded A^{WH}-modules, and it is therefore free as well, since finitely generated graded projective modules are free by Nakayama's lemma for connected graded algebras. Hence

$$A^H \simeq A^{WH}(\operatorname{Ind}_H^{WH} k)$$

is a free graded A^{WH}-module. In particular, A^{WH} is a direct summand of A^H as graded A^{WH}-module, and therefore also as graded A^G-module. Since A^H is free over A^G, it follows again that A^{WH} is also free over A^G.

In the remaining part of the proof of (i), we shall use the basic properties of the Noether different $\mathfrak{D}^N$ (also called homological different) and the Dedekind different $\mathfrak{D}^D$ of the extension, cf. [3] or [9]. By a theorem of Noether and Auslander-Buchsbaum, see [3, Theorem 3.11.1] or [9, Lemma 2(iii)], freeness implies that

$$\mathfrak{D}^N_{A^{WH}/A^G} = \mathfrak{D}^D_{A^{WH}/A^G}.$$

Since $\mathfrak{D}^D_{A^{WH}/A^G} = (1)$, we get $\mathfrak{D}^N_{A^{WH}/A^G} = (1)$. This means that 1 annihilates the kernel of the multiplication map

$$A^{WH} \otimes_{A^G} A^{WH} \to A^{WH}.$$

Hence it is an isomorphism. Now A^{WH} is free over A^G, say of rank r. Therefore $A^{WH} \simeq A^{WH} \otimes_{A^G} A^{WH}$ has rank r over A^{WH}, hence $r = 1$. We conclude that $A^G = A^{WH}$ and so from Galois theory it follows that indeed $G = WH$.

(ii) Since $G = WH = KH$, it follows that $\operatorname{Ind}_H^G k = \operatorname{Ind}_{K\cap H}^K k$ as kK-modules. Now we can apply Theorem 2 and using that

$$A^H \simeq A^G(\operatorname{Ind}_H^G k) \text{ and } A^{K\cap H} \simeq A^K(\operatorname{Ind}_{K\cap H}^K k)$$

we obtain the isomorphism

$$\mu : A^K \otimes_{A^G} A^H \simeq A^{K\cap H}.$$

From this the remaining assertions in (ii) and (iii) follow easily. □

In the linear case, we also have a converse.

Proposition 8. *Let the finite group G act linearly on V and let H and K be subgroups such that $W \leq K \leq G$. Suppose $G = KH$, $k[V]^{K\cap H}$ is free over $k[V]^K$ and $k[V]^{K\cap H}$ is generated by $k[V]^K$ and $k[V]^H$. Then $k[V]^H$ is free over $k[V]^G$.*

Proof. Since $k[V]^{K\cap H}$ is generated by $k[V]^K$ and $k[V]^H$, the $k[V]^K$-module homomorphism

$$\mu : k[V]^K \otimes_{k[V]^G} k[V]^H \to k[V]^{K\cap H}$$

induced by multiplication is surjective. So the $k[V]^G$-module $k[V]^H$ and the $k[V]^K$-module $k[V]^{K\cap H}$ share generators. Since $G = KH$, we get that $\operatorname{Ind}_H^G k$ and $\operatorname{Ind}_{K\cap H}^K k$ are isomorphic as kK-modules. Hence

$$k[V]^{K\cap H} \simeq k[V]^K(\operatorname{Ind}_{K\cap H}^K k) \simeq k[V]^K(\operatorname{Ind}_H^G k)$$

as $k[V]^K$-modules. And

$$k[V]^G(\operatorname{Ind}_H^G k) \simeq k[V]^H$$

as $k[V]^G$-modules. So $k[V]^K(\operatorname{Ind}_H^G k)$ is free and shares generators with $k[V]^G(\operatorname{Ind}_H^G k)$. By Corollary 1, we conclude that $k[V]^G(\operatorname{Ind}_H^G k) = k[V]^H$ is free over $k[V]^G$. □

4.2. **Examples of free extensions.** In the linear case, Chevalley-Shephard-Todd's classical theorem says that if G is generated by reflections and G is non-modular, then the extension $k[V]^G \subset k[V]$ is a graded complete intersection extension, cf. [3, Theorem 7.2.1], [9, Theorem 6]. This was generalized by Hochster-Eagon [18] to more general actions by adapting Chevalley's conceptual proof. An automorphism σ of an integral domain R is called a *Hochster-Eagon reflection* if there is a non-zero $f \in R$ such that $\sigma(a) - a \in (f)$ for all $a \in R$. Hochster-Eagon reflections are also reflections in our sense, but not necessarily conversely. If R is factorial, then reflections are the same as Hochster-Eagon reflections.

Proposition 9 (Hochster-Eagon, Avramov)**.** *Let A be a normal graded algebra and G a finite group of graded algebra automorphisms of A. Suppose $H \lhd G$, G/H is non-modular and the action of G/H on A^H is generated by Hochster-Eagon reflections. Then $A^G \subset A^H$ is a graded complete intersection extension.*

Proof. Hochster-Eagon [18] proved the freeness of $A^G \subset A^H$, and Avramov [2] noticed that the fiber algebra is a graded complete intersection. So, in other words, $A^G \subset A^H$ is a graded complete intersection extension. □

Example 5. An example of a graded complete intersection extension of a different kind is due to Nakajima [23]. Recall the definition of G^1 in subsection 1.1.

Proposition 10 (Nakajima). *(i) Suppose that A^G is factorial and $G^1 \leq H \leq G$. Then $A^G \subset A^H$ is a free extension and G/H is non-modular.*

(ii) Suppose that A is factorial and G is generated by reflections on A. Then $A^G \subset A^{G^1}$ is a graded complete intersection extension.

Proof. These results were (implicitly) proved by Nakajima [23]. For (i), we note that A^H is isomorphic to the direct sum of all modules of semi-invariants of types λ having H in the kernel. Any module of semi-invariants is reflexive of rank one, hence isomorphic to a divisorial ideal. Since A^G is factorial by assumption, all divisorial ideals are principal ideals. So all modules of semi-invariants are free and so $k[V]^G \subset k[V]^H$ is free. For (ii), we refer to Nakajima [23]. □

Example 6. An example of a modular free extension. Suppose A^H is factorial, $HW = G$ and $|G/H| = p$ (the characteristic of k). Then A^G is also factorial, and $A^G \subset A^H$ is a graded complete intersection extension *if and only if* A^G is a direct summand of A^H, cf. [10]. The direct summand property holds, for example, when $A^G \subset A$ is a free graded extension.

5. A ramification formula

Let A be a normal graded algebra with a finite group G of k-algebra automorphisms, M a finite dimensional kG-module, and $H < G$ a subgroup. We use the transfer map

$$\mathrm{Tr}^H : A \otimes_k M \to (A \otimes_k M)^H : u \mapsto \sum_{\sigma \in H} \sigma(u)$$

and the relative transfer map

$$\mathrm{Tr}^G_H : (A \otimes_k M)^H \to (A \otimes_k M)^G : u \mapsto \sum_{\sigma H \in G/H} \sigma(u)$$

to construct the following homomorphisms:

$$\Phi^H : A \otimes_k M \to \mathrm{Hom}_{A^H}(A, (A \otimes_k M)^H) : \ [\Phi^H(\omega)](a) := \mathrm{Tr}^H(a\omega)$$

where $\omega \in A \otimes_k M$, $a \in A$ and

$$\Phi^G_H : \mathrm{Hom}_{A^H}(A, (A \otimes_k M)^H) \to \mathrm{Hom}_{A^G}(A, (A \otimes_k M)^G) : \ \phi \mapsto \mathrm{Tr}^G_H \circ \phi.$$

Both Φ^H and Φ^G_H are injective A-module homomorphisms between reflexive A-modules and

$$\Phi^G = \Phi^G_H \circ \Phi^H.$$

The following theorem is the technical heart of this article; all our main results rely on it.

Theorem 5. *Let A be a normal graded algebra with a finite group G of k-algebra automorphisms, M a finite dimensional kG-module, and $H < G$ a subgroup.*

(i) The A-module homomorphism of reflexive A-modules

$$\Phi^G_H : \mathrm{Hom}_{A^H}(A, (A \otimes_k M)^H) \to \mathrm{Hom}_{A^G}(A, (A \otimes_k M)^G)$$

is injective, generically an isomorphism, and an isomorphism at the prime ideal of height one $\mathfrak{P} \subset A$ if we have an equality of inertia subgroups $G_i(\mathfrak{P}) = H_i(\mathfrak{P})$.

(ii) For any subgroup $K < G$ containing all reflections in G on A, the map Φ_K^G is an isomorphism.

(iii) Write C_H^G for the cokernel of Φ_H^G. Let $\mathfrak{P}$ be a height one prime ideal of A with inertia subgroups $H_i := H_i(\mathfrak{P})$ and $G_i := G_i(\mathfrak{P})$. We have $\left(C_H^G\right)_{\mathfrak{P}} \simeq \left(C_{H_i}^{G_i}\right)_{\mathfrak{P}}$, and so in particular,

$$\ell_{A_{\mathfrak{P}}}\left(\left(C_H^G\right)_{\mathfrak{P}}\right) = \ell_{A_{\mathfrak{P}}}\left(\left(C_{H_i}^{G_i}\right)_{\mathfrak{P}}\right).$$

Proof. Fix a prime ideal $\mathfrak{P} \subset A$ with decomposition group G_d and inertia subgroup G_i. Put

$$\mathfrak{p} := \mathfrak{P} \cap A^G,\ \mathfrak{P}_i := A^{G_i} \cap \mathfrak{P},\ \mathfrak{P}_d := A^{G_d} \cap \mathfrak{P},\ R := (A^{G_d})_{\mathfrak{P}_d},\ S := (A^{G_i})_{\mathfrak{P}_i}$$

and $\Gamma := G_d/G_i$. It is known that $\mathfrak{P}$ is the only prime ideal of A lying above $\mathfrak{P}_d$ (or $\mathfrak{P}_i$) (see e.g. [29, Satz 20.4]). For any A-module N, we get $N_{\mathfrak{P}_d} \simeq N_{\mathfrak{P}_i} \simeq N_{\mathfrak{P}}$, and $(A^{G_i})_{\mathfrak{P}_d} \simeq (A^{G_i})_{\mathfrak{P}_i} = S$.

If B is a flat A^G-algebra with trivial G-action, then $(A \otimes_{A^G} B)^G \simeq A^G \otimes_{A^G} B \simeq B$, and similarly, $((A \otimes_{A^G} B) \otimes_k M)^G \simeq (A \otimes_k M)^G \otimes_{A^G} B$, cf. [19, Lemma 1]. In particular, $S \simeq (A_{\mathfrak{P}_i})^{G_i} \simeq (A_{\mathfrak{P}})^{G_i}$ and $R \simeq (A_{\mathfrak{P}_d})^{G_d} \simeq (A_{\mathfrak{P}})^{G_d}$. We shall use this repeatedly without further mention.

We proceed in various steps.

(1) In the first step, we prove that $\Phi_{G_i}^{G_d}$ is an isomorphism if $\mathfrak{P}$ has height one or if $\mathfrak{P} = (0)$. Furthermore, we prove that Φ_H^G is injective and generically an isomorphism for any subgroup $H < G$.

Some preparations first. The extension $R \subset S$ is a Galois-extension of local rings with Galois group Γ (see e.g. [29, Satz 20.4]). In particular, S is free over $R = S^\Gamma$ of finite rank;

$$S \to \mathrm{Hom}_R(S, R) : s_1 \mapsto (s_2 \mapsto \mathrm{Tr}^\Gamma(s_1 s_2)) \tag{5}$$

is an isomorphism of S-modules (see [29, 12.5 Korollar] or [1, Appendix]); and the natural inclusion $S\Gamma \to \mathrm{End}_R(S)$ is an isomorphism of R-algebras, where $S\Gamma$ is the twisted group ring. The twisted group ring $S\Gamma$ is free as left S-module with basis $\{\gamma;\ \gamma \in \Gamma\}$ and multiplication table $(s_1\gamma_1)(s_2\gamma_2) = s_1\gamma_1(s_2)\gamma_1\gamma_2$, where $s_1, s_2 \in S$, $\gamma_1, \gamma_2 \in \Gamma$. Fix an R-basis $z_1, \ldots, z_n$ of S, with dual basis $z_1^*, \ldots, z_n^*$ of $\mathrm{Hom}_R(S, R)$. There are unique u_i's such that $z_i^*(s) = \mathrm{Tr}^\Gamma(u_i s)$, for $s \in S$. We get an equality of operators $\sum_i z_i \mathrm{Tr}^\Gamma u_i = 1$ in $S\Gamma$.

Put $N := (A_{\mathfrak{P}} \otimes_k M)^{G_i}$. We claim that the multiplication map

$$\mu : S \otimes_R N^\Gamma \to N$$

is an isomorphism of $S\Gamma$-modules, where $S\Gamma$ only acts non-trivially on the first factor of $S \otimes_R N^\Gamma$. Its inverse is $\alpha : N \to S \otimes_R N^\Gamma$ given by the formula

$$\alpha(n) := \sum_i z_i \otimes \mathrm{Tr}^\Gamma(u_i n),$$

where $n \in N$. We check this as follows. For $n \in N$, we have $\mu\alpha(n) = \sum_i z_i \operatorname{Tr}^\Gamma(u_i n) = n$, since $\sum_i z_i \operatorname{Tr}^\Gamma u_i = 1 \in S\Gamma$. Also

$$\alpha(\mu(s \otimes n)) = \sum_i z_i \otimes \operatorname{Tr}^\Gamma(u_i s n) = \sum_i z_i \otimes \operatorname{Tr}^\Gamma(u_i s)n = \sum_i z_i \operatorname{Tr}^\Gamma(u_i s) \otimes n = s \otimes n,$$

where $s \in S$, $n \in N^\Gamma$.

The localization of $\Phi_{G_i}^{G_d}$ at the prime ideal $\mathfrak{P}$ is described as follows. Since

$$\big(\operatorname{Hom}_{A^{G_d}}(A, (A \otimes_k M)^{G_d})\big)_{\mathfrak{P}_d} \simeq \operatorname{Hom}_{A^{G_d}_{\mathfrak{P}_d}}(A_{\mathfrak{P}_d}, (A \otimes_k M)^{G_d}_{\mathfrak{P}_d}) \simeq \operatorname{Hom}_R(A_{\mathfrak{P}}, (A_{\mathfrak{P}} \otimes_k M)^{G_d}),$$

and similarly $\big(\operatorname{Hom}_{A^{G_i}}(A, (A \otimes_k M)^{G_i})\big)_{\mathfrak{P}_d} \simeq \operatorname{Hom}_S(A_{\mathfrak{P}}, (A_{\mathfrak{P}} \otimes_k M)^{G_i})$, we obtain

$$\left(\Phi_{G_i}^{G_d}\right)_{\mathfrak{P}} : \operatorname{Hom}_S(A_{\mathfrak{P}}, (A_{\mathfrak{P}} \otimes_k M)^{G_i}) \to \operatorname{Hom}_R(A_{\mathfrak{P}}, (A_{\mathfrak{P}} \otimes_k M)^{G_d}) : \phi \mapsto \operatorname{Tr}_{G_i}^{G_d} \circ \phi.$$

Using the isomorphism μ, we finally get the following description

$$\left(\Phi_{G_i}^{G_d}\right)_{\mathfrak{P}} : \operatorname{Hom}_S(A_{\mathfrak{P}}, S \otimes_R N^\Gamma) \to \operatorname{Hom}_R(A_{\mathfrak{P}}, N^\Gamma) : \phi \mapsto \operatorname{Tr}^\Gamma \circ \mu \circ \phi.$$

The normality condition on A implies that if $\mathfrak{P}$ has height one, then R and S are both discrete valuation rings. So $A_{\mathfrak{P}}$ is finite and free over S and N^Γ is finite and free over R, since both modules are torsion free. Therefore, to prove that $\Phi_{G_i}^{G_d}$ is an isomorphism at the height one prime ideal $\mathfrak{P}$, it suffices to prove that

$$\operatorname{Hom}_S(S, S) \to \operatorname{Hom}_R(S, R) : \phi \mapsto \operatorname{Tr}^\Gamma$$

is an isomorphism. But this follows from the isomorphism in (5).

A similar proof works for $\mathfrak{P} = (0)$. In that case, $G = G_d$ and $G_i = 1$ and so, with L the quotient field of A, we conclude

$$(\Phi^G)_{\mathfrak{P}} : L \otimes_k M \to \operatorname{Hom}_{L^G}(L, (L \otimes_k M)^G)$$

is an isomorphism. Since for any subgroup $H < G$, also $(\Phi^H)_{\mathfrak{P}}$ is an isomorphism and $\Phi^G = \Phi_H^G \circ \Phi^H$, it follows that $(\Phi_H^G)_{\mathfrak{P}}$ is also an isomorphism, i.e., Φ_H^G is generically an isomorphism.

Finally, we prove injectivity of Φ_H^G. Let $\phi : A \to (A \otimes_k M)^H$ be an A^H-homomorphism, such that $\operatorname{Tr}_H^G \circ \phi(a) = 0$, for all $a \in A$. We extend ϕ to an L^H-linear map $\tilde{\phi} : L \to (L \otimes_k M)^H$ by $\tilde{\phi}(\frac{a}{s}) = \frac{1}{s}\phi(a)$, where $a \in A$, $s \in A^H$, $s \neq 0$. So $\tilde{\phi}$ is in the kernel of the map

$$\operatorname{Hom}_{L^H}(L, (L \otimes_k M)^H) \to \operatorname{Hom}_{L^G}(L, (L \otimes_k M)^G) : \psi \to \operatorname{Tr}_H^G \circ \psi,$$

which we just proved to be an isomorphism. So $\tilde{\phi} = 0$, hence $\phi = 0$ (since $(A \otimes_k M)^H$ is torsion free over A^H) and so Φ_H^G is injective.

(2) In the second step, we show that $\Phi_{G_d}^G$ is an isomorphism at $\mathfrak{P}$ (without any restriction on the height of $\mathfrak{P}$), using completion as a tool.

Some preparations first. For any A^G-module N, write $\widehat{N} = N^\wedge$ for the completion of the localization $N_{\mathfrak{p}}$ with respect to the $\mathfrak{p}$-adic topology. So $\widehat{N}$ is a module over the complete local

ring $\widehat{A^G}$, the completion of the local ring $A^G_{\mathfrak{p}}$ at its maximal ideal. We recall the following basic facts about completion. Putting the hat on is an exact functor from A^G-modules to $\widehat{A^G}$-modules, and so $\widehat{A}$ is flat over $\widehat{A^G} \simeq \widehat{A}^G$. The ring $\widehat{A}$ is a complete semilocal ring, whose maximal ideals correspond to the prime ideals of A in the G-orbit of $\mathfrak{P}$, see [13, Corollary 7.6]. More explicitly, there is a collection of idempotents $\{e_{g\mathfrak{P}}, gG_d \in G/G_d\}$ in $\widehat{A}$, such that $g(e_{\mathfrak{P}}) = e_{g\mathfrak{P}}$ for $g \in G$ and

$$e_{g\mathfrak{P}}^2 = e_{g\mathfrak{P}},\ e_{g\mathfrak{P}} \cdot e_{g'\mathfrak{P}} = 0 \text{ if } g\mathfrak{P} \neq g'\mathfrak{P},\ 1 = \sum_{gG_d \in G/G_d} e_{g\mathfrak{P}} \in \widehat{A}.$$

The Cartesian product alluded to before is then $\widehat{A} = \bigoplus_{gG_d \in G/G_d} \widehat{A} e_{g\mathfrak{P}}$. The maximal ideal of $\widehat{A}$ corresponding to $\mathfrak{P}$ is

$$\mathfrak{m} := \widehat{\mathfrak{p}} \widehat{A} e_{\mathfrak{P}} \oplus \widehat{A}(1 - e_{\mathfrak{P}}) = \widehat{\mathfrak{P}} = \mathfrak{P}\widehat{A},$$

and $g(\mathfrak{m})$ is the maximal ideal corresponding to $g\mathfrak{P}$, for $g \in G$; and these are all the maximal ideals of $\widehat{A}$. Then $\widehat{A} e_{\mathfrak{P}} \simeq \widehat{A}_{\mathfrak{m}} \simeq \widehat{A_{\mathfrak{P}}}$ is isomorphic to the completion of $A_{\mathfrak{P}}$ at its maximal ideal. Similarly, for any A-module N we have that $e_{\mathfrak{P}}\widehat{N} \simeq \widehat{N}_{\mathfrak{m}} \simeq \widehat{N_{\mathfrak{P}}}$ is isomorphic to the completion of $N_{\mathfrak{P}}$ at the maximal ideal of $A_{\mathfrak{P}}$.

We apply all this to the A-module

$$N := \mathrm{Hom}_{A^{G_d}}(A, (A \otimes_k M)^{G_d}).$$

First of all, we can identify $\widehat{N}$ with $\mathrm{Hom}_{\widehat{A}^{G_d}}(\widehat{A}, (\widehat{A} \otimes_k M)^{G_d})$. Let $\phi \in \widehat{N}$. By definition of the $\widehat{A}$-action on $\widehat{N}$, the element $e_{\mathfrak{P}}\phi \in e_{\mathfrak{P}}\widehat{N}$ is the $\widehat{A}^{G_d}$-morphism $(e_{\mathfrak{P}}\phi)(a) := \phi(a e_{\mathfrak{P}})$, where $a \in \widehat{A}$. Since $e_{\mathfrak{P}}$ is G_d-invariant and ϕ is $\widehat{A}^{G_d}$-linear, we also have

$$(e_{\mathfrak{P}}\phi)(a) = \phi(a e_{\mathfrak{P}}) = \phi(a e_{\mathfrak{P}} e_{\mathfrak{P}}) = e_{\mathfrak{P}}\phi(a e_{\mathfrak{P}});$$

and so the image of $e_{\mathfrak{P}}\phi$ is contained in $(\widehat{A} e_{\mathfrak{P}} \otimes_k M)^{G_d}$. Therefore, we can make the identification

$$e_{\mathfrak{P}}\widehat{N} \simeq \mathrm{Hom}_{\widehat{A}^{G_d}}(\widehat{A} e_{\mathfrak{P}}, (\widehat{A} e_{\mathfrak{P}} \otimes_k M)^{G_d}).$$

Similarly, we identify

$$e_{\mathfrak{P}} \left(\mathrm{Hom}_{A^G}(A, (A \otimes_k M)^G)\right)^{\wedge} \simeq \mathrm{Hom}_{\widehat{A}^{G_d}}(\widehat{A} e_{\mathfrak{P}}, (\widehat{A} \otimes_k M)^{G_d}),$$

and then identify $e_{\mathfrak{P}} \cdot \left(\Phi^G_{G_d}\right)^{\wedge}$ with the map

$$\mathrm{Hom}_{\widehat{A}^{G_d}}(\widehat{A} e_{\mathfrak{P}}, (\widehat{A} e_{\mathfrak{P}} \otimes_k M)^{G_d}) \to \mathrm{Hom}_{\widehat{A}^G}(\widehat{A} e_{\mathfrak{P}}, (\widehat{A} \otimes_k M)^G) :\ \phi \mapsto \mathrm{Tr}^G_{G_d} \circ \phi. \tag{6}$$

We want to show that this map is an isomorphism.

As a preparation of the proof, we first remark that, after $\mathfrak{p}$-adic completion, the natural map $A^G_{\mathfrak{p}} \to (A_{\mathfrak{P}})^{G_d}$ can be identified with

$$\pi : \widehat{A}^G \to (\widehat{A} e_{\mathfrak{P}})^{G_d};\ \pi(a) := a e_{\mathfrak{P}}.$$

We claim that π is an isomorphism of algebras, with inverse map given by the relative trace map $\operatorname{Tr}^G_{G_d}$. This is seen as follows. Let $a \in \widehat{A}$ and $\sigma \in G$, then we can write

$$a = \sum_{gG_d \in G/G_d} ae_{g\mathfrak{P}} \text{ and so, } \sigma(a) = \sum_{gG_d \in G/G_d} \sigma(a) e_{\sigma g \mathfrak{P}}.$$

Comparing, we get that $a \in \widehat{A}^G$ *if and only if* $ae_{\mathfrak{P}} \in (\widehat{A}e_{\mathfrak{P}})^{G_d}$ and $ae_{g\mathfrak{P}} = g(ae_{\mathfrak{P}})$ for all $gG_d \in G/G_d$ *if and only if* $a = \operatorname{Tr}^G_{G_d}(ae_{\mathfrak{P}}) = \operatorname{Tr}^G_{G_d}\pi(a)$. Hence, indeed, π is an algebra isomorphism with inverse $\operatorname{Tr}^G_{G_d}$.

Similarly, after $\mathfrak{p}$-adic completion, the natural map $(A_{\mathfrak{p}} \otimes_k M)^G \to (A_{\mathfrak{P}} \otimes_k M)^{G_d}$ can be identified with the map

$$\pi_M : (\widehat{A} \otimes_k M)^G \to (\widehat{A}e_{\mathfrak{P}} \otimes_k M)^{G_d}$$

induced by projection, and, again, π_M is an isomorphism with inverse map $\operatorname{Tr}^G_{G_d}$.

Let now $\psi \in \operatorname{Hom}_{\widehat{A}^G}(\widehat{A}e_{\mathfrak{P}}, (\widehat{A} \otimes_k M)^G)$ and consider the composition

$$\pi_M \circ \psi : \widehat{A}e_{\mathfrak{P}} \to (\widehat{A}e_{\mathfrak{P}} \otimes_k M)^G).$$

Let $b \in \widehat{A}^{G_d}$ and $ae_{\mathfrak{P}} \in \widehat{A}e_{\mathfrak{P}}$. Then $\operatorname{Tr}^G_{G_d}(be_{\mathfrak{P}}) \in \widehat{A}^G$ and so

$$\pi_M \circ \psi(bae_{\mathfrak{P}}) = e_{\mathfrak{P}}\psi(\operatorname{Tr}^G_{G_d}(be_{\mathfrak{P}})ae_{\mathfrak{P}}) = e_{\mathfrak{P}} \operatorname{Tr}^G_{G_d}(be_{\mathfrak{P}})\psi(ae_{\mathfrak{P}}) = b(\pi_M \circ \psi)(bae_{\mathfrak{P}}),$$

using the orthogonality of the idempotents and the $\overline{A}^G$-linearity of ψ. We conclude that $\pi_M \circ \psi$ is $\widehat{A}^{G_d}$-linear. We get, therefore, a well-defined map

$$\Psi : \operatorname{Hom}_{\widehat{A}^G}(\widehat{A}e_{\mathfrak{P}}, (\widehat{A} \otimes_k M)^G) \to \operatorname{Hom}_{\widehat{A}^{G_d}}(\widehat{A}e_{\mathfrak{P}}, (\widehat{A}e_{\mathfrak{P}} \otimes_k M)^{G_d}) : \ \psi \mapsto \pi_M \circ \psi,$$

which is the inverse of the map $e_{\mathfrak{P}} \cdot \left(\Phi^G_{G_d}\right)^\wedge$ in (6), since $\operatorname{Tr}^G_{G_d} \circ \pi_M$ and $\pi_M \circ \operatorname{Tr}^G_{G_d}$ are the identity maps.

Since $e_{\mathfrak{P}} \cdot \left(\Phi^G_{G_d}\right)^\wedge$ can be identified with what we get from $\Phi^G_{G_d}$ after localizing at $\mathfrak{P}$ and then completing at $\mathfrak{P}$, we conclude that, indeed, $\Phi^G_{G_d}$ becomes an isomorphism after localizing at $\mathfrak{P}$ and then completing with respect to the $\mathfrak{P}$-adic topology. But then $\Phi^G_{G_d}$ already becomes an isomorphism after localizing at the prime ideal $\mathfrak{P}$, see [13, p. 203]. Which is what we wanted to show.

(3) In the last step, we complete the proof of the theorem. That Φ^G_H is indeed a homomorphism of reflexive A-modules is implied by Lemma 4(ii), (iv).

Combining (1) and (2), we get that $\Phi^G_{G_i}$ is an isomorphism at every prime ideal $\mathfrak{P}$ of height one. Suppose $H < G$ is a subgroup and $\mathfrak{P} \subset A$ a prime ideal of height one, such that the inertia subgroups coincide $H_i(\mathfrak{P}) = G_i(\mathfrak{P})$. Since

$$\Phi^G_H \circ \Phi^H_{H_i(\mathfrak{P})} = \Phi^G_{G_i(\mathfrak{P})}$$

and both $\Phi^H_{H_i(\mathfrak{P})}$ and $\Phi^G_{G_i(\mathfrak{P})}$ are isomorphisms at $\mathfrak{P}$ by (2), it follows that Φ^G_H is an isomorphism at $\mathfrak{P}$. Hence (i). And from this also the isomorphism $\left(C^G_H\right)_{\mathfrak{P}} \simeq \left(C^{G_i}_{H_i}\right)_{\mathfrak{P}}$ follows, hence (iii). The condition in (ii) implies that $G_i(\mathfrak{P}) = K_i(\mathfrak{P})$ for all height one prime ideals.

So Φ_K^G is a pseudo-isomorphism between reflexive modules and thus is an isomorphism, by Lemma 4. Hence (ii). □

5.1. **Proof of Theorem 1.** The following is a generalization of a result of Benson and Crawley-Boevey [3, Corollary 3.12.2] (we formulated this result earlier as Theorem 1).

Corollary 2. *Let A be a normal graded algebra with a finite group G of k-algebra automorphisms, and M a finite dimensional kG-module.*

(i) For any subgroup $K \leq G$ such that $K \supseteq W$ we have

$$s_{A^G}(A^G(M)) = s_{A^K}(A^K(M)).$$

In particular, if W acts trivially on M, then

$$s_{A^G}(A^G(M)) = 0.$$

(ii) Suppose $G_i(\mathfrak{P}) \cap G_i(\mathfrak{P}') = \{1\}$ for distinct height one prime ideals $\mathfrak{P}$ and $\mathfrak{P}'$ of A. Then

$$s_{A^G}(A^G(M)) = \sum_{\mathfrak{P}} s_{A^{G_i(\mathfrak{P})}}(A^{G_i(\mathfrak{P})}(M))$$

where the sum is over the homogeneous height one prime ideals of A.

Proof. (i) From Theorem 5, we get an isomorphism of A-modules

$$\mathrm{Hom}_{A^K}(A, A^K(M)) \simeq \mathrm{Hom}_{A^G}(A, A^G(M)).$$

We calculate using Proposition 1:

$$s_{A^G}(\mathrm{Hom}_{A^G}(A, A^G(M))) = |G| s_{A^G}(A^G(M)) - \dim_k M \cdot s_{A^G}(A),$$

since $s_{A^G}(\mathrm{Ext}^1_{A^G}(A, A^G(M))) = 0$ (because A is A^G-torsion free and A^G is normal). On the other hand, Lemma 1 and Proposition 1 give

$$\begin{aligned} & s_{A^G}(\mathrm{Hom}_{A^K}(A, A^K(M))) = \\ = \quad & s_{A^K}(\mathrm{Hom}_{A^K}(A, A^K(M)))\frac{|G|}{|K|} + |K| \dim_k M \cdot s_{A^G}(A^K) \\ = \quad & |G| s_{A^K}(A^K(M)) - \dim_k M \cdot s_{A^K}(A)\frac{|G|}{|K|}, \end{aligned}$$

because $s_{A^G}(A^K) = 0$, by Proposition 2. Since

$$s_{A^G}(A) = s_{A^K}(A)\frac{|G|}{|K|} + |K| s_{A^G}(A^K) = s_{A^K}(A)\frac{|G|}{|K|},$$

we get that

$$s_{A^G}(A^G(M)) = s_{A^K}(A^K(M)).$$

(ii) Let $\mathfrak{P}$ be a homogeneous height one prime ideal of A with inertia subgroup $H := G_i(\mathfrak{P})$. Let $\mathfrak{P}'$ be another homogeneous height one prime ideal of A. Then by the assumption $H_i(\mathfrak{P}') = 1$. So the cokernel $C_{\mathfrak{P}}$ of

$$\Phi^H : A \otimes_k M \to \operatorname{Hom}_{A^H}(A, A^H(M))$$

vanishes at $\mathfrak{P}'$. Therefore, since Φ^H is injective, we get

$$\ell_{\mathfrak{P}}(C_{\mathfrak{P}})\psi(A/\mathfrak{P}) = \psi(\operatorname{Hom}_{A^H}(A, A^H(M))) - \psi(A)\dim_k M.$$

So, from the theorem, it follows that

$$\begin{aligned} &\psi(\operatorname{Hom}_{A^G}(A, A^G(M))) - \psi(A)\dim_k M = \\ &= \sum_{\mathfrak{P}} \left(\psi(\operatorname{Hom}_{A^{G_i(\mathfrak{P})}}(A, A^{G_i(\mathfrak{P})}(M))) - \psi(A)\dim_k M\right), \end{aligned}$$

where the sum is over the homogeneous height one prime ideals of A.

A direct calculation as in (i) shows that

$$\psi(\operatorname{Hom}_{A^G}(A, A^G(M))) - \psi(A)\dim_k M = -\deg(A)s_{A^G}(A^G(M)) + 2\deg(A)\dim_k M \frac{s_{A^G}(A)}{|G|}$$

and similarly, where G is replaced by $G_i(\mathfrak{P})$. Using Proposition 2(ii) that

$$\frac{1}{|G|}s_{A^G}(A) = \sum_{\mathfrak{P}} \frac{1}{|G_i(\mathfrak{P})|}s_{A^{G_i(\mathfrak{P})}}(A),$$

we get

$$s_{A^G}(A^G(M)) = \sum_{\mathfrak{P}} s_{A^{G_i(\mathfrak{P})}}(A^{G_i(\mathfrak{P})}(M)),$$

where both sums are over the homogeneous height one prime ideals of A. □

The following corollary (of the proof of Theorem 5) is used in the proof of Proposition 7.

Lemma 6. *Suppose $A^G(M)$ is free (or Cohen-Macaulay), and $\mathfrak{P} \subset A$ a prime ideal with decomposition group G_d and inertia group G_i. Then*

(i) $A^{G_d}(M)$ is free (or Cohen-Macaulay) at the prime ideal $\mathfrak{P} \cap k[V]^{G_d}$.

(ii) $A^{G_i}(M)$ is free (or Cohen-Macaulay) at the prime ideal $\mathfrak{P} \cap k[V]^{G_i}$.

Proof. We use the results, techniques and notation of the proof of Theorem 5.

(i) We proved that

$$(\widehat{A}e_{\mathfrak{P}} \otimes_k M)^{G_d} \simeq (\widehat{A} \otimes_k M)^G \simeq ((A \otimes_k M)^G)^\wedge,$$

and so $(\widehat{A}e_{\mathfrak{P}} \otimes_k M)^{G_d}$ is free (or Cohen-Macaulay) over $\widehat{A}^G \simeq (\widehat{A}e_{\mathfrak{P}})^{G_d}$. On the other hand, $\widehat{A}e_{\mathfrak{P}}$ is also the completion of $A_{\mathfrak{P}_d}$ with respect to the $\mathfrak{P}_d$-adic topology. So the completion of $A^{G_d}(M)_{\mathfrak{P}_d}$ with respect to the $\mathfrak{P}_d$-adic topology is free (or Cohen-Macaulay), hence $A^{G_d}(M)_{\mathfrak{P}_d}$ is free (or Cohen-Macaulay) over $A^{G_d}_{\mathfrak{P}_d}$.

(ii) We found that multiplication induces an isomorphism of $S\Gamma$-modules

$$S \otimes_R (A_{\mathfrak{P}} \otimes_k M)^{G_d} \simeq (A_{\mathfrak{P}} \otimes_k M)^{G_i}.$$

Since we proved in (i) that $(A_{\mathfrak{P}} \otimes_k M)^{G_d}$ is free over R (or Cohen-Macaulay) it follows that $(A_{\mathfrak{P}} \otimes_k M)^{G_i}$ is also free (or Cohen-Macaulay), since S is free over R). $\square$

Remark. Reformulating the corollary in terms of the numerical invariant ψ we get for any subgroup $W \leq K \leq G$ that

$$|G|\psi(A^G(M)) = |K|\psi(A^K(M)).$$

In particular, when $W = 1$, then

$$|G|\psi(A^G(M)) = \psi(A)\dim_k M.$$

And if $G_i(\mathfrak{P}) \cap G_i(\mathfrak{P}') = \{1\}$, for distinct height one prime ideals $\mathfrak{P}$ and $\mathfrak{P}'$ of A, then

$$|G|\psi(A^G(M)) - \dim_k M\psi(A) = \sum_{\mathfrak{P}} \left(|G_i(\mathfrak{P})|\psi(A^{G_i(\mathfrak{P})}(M)) - \dim_k M\psi(A)\right),$$

where the sum is over the homogeneous height one prime ideals of A.

Example 7. Take the special case where $M = kG$ is the regular representation. If we restrict this representation to a subgroup H, it decomposes as the direct sum of $\frac{|G|}{|H|}$ copies of the regular representation kH of H. So under the hypothesis of Corollary 2(ii) we get

$$\begin{aligned} s_{A^G}(A) &= s_{A^G}(A^G(kG)) \\ &= \sum_{\mathfrak{P}} s_{A^{G_i(\mathfrak{P})}}(A^{G_i(\mathfrak{P})}(kG)) \\ &= \sum_{\mathfrak{P}} \frac{|G|}{|G_i(\mathfrak{P})|} s_{A^{G_i(\mathfrak{P})}}(A^{G_i(\mathfrak{P})}(kG_i(\mathfrak{P}))) \\ &= \sum_{\mathfrak{P}} \frac{|G|}{|G_i(\mathfrak{P})|} s_{A^{G_i(\mathfrak{P})}}(A) \end{aligned}$$

where the sum is over the homogeneous height one prime ideals of A. Hence we recover Proposition 2(ii).

5.2. **Proof of the Jacobian criterion.** We shall use the techniques of the proof of Theorem 5 to give a proof of the Jacobian criterion of freeness of modules of covariants, i.e., Theorem 3.

Proof of Theorem 3. Write $A := k[V]$. Since Jac_M is multilinear in its arguments, we get an A^G-linear map

$$\mathrm{Jac}_M : \wedge^m_{A^G} (A \otimes_k M)^G \to A : \ \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_m \mapsto \mathrm{Jac}_M(\omega_1, \ldots, \omega_m).$$

Its image is just J_M^G, an A^G-submodule of A contained in A_λ^G. Let $A^G \subset B$ be a flat extension of algebras, then

$$\begin{aligned} B \otimes_{A^G} \wedge^m_{A^G} (A \otimes_k M)^G &\simeq \wedge^m_B \left(B \otimes_{A^G} (A \otimes_k M)^G \right) \\ &\simeq \wedge^m_B (B \otimes_{A^G} A \otimes_k M)^G , \end{aligned}$$

by [13, p. 571] and our remark at the beginning of the proof of Theorem 5. The corresponding Jacobian map

$$B \otimes_{A^G} \mathrm{Jac}_M : \wedge^m_B (B \otimes_{A^G} A \otimes_k M)^G \to B \otimes_{A^G} A$$

has then image $B \otimes_{A^G} J_M^G \subset B \otimes_{A^G} A$.

Let $\mathfrak{P} \subset A$ be a prime ideal. We use the results, techniques and notation of the proof of Theorem 5. In particular, $\mathfrak{p}$, G_i, G_d, $\widehat{A}$, $e_{\mathfrak{P}}$, $\mathfrak{m}$, etc. We will break the proof in several steps. That the action on $A := k[V]$ comes from a linear action on V does not play a role in the first three steps.

(1) In the first step, we prove that $J_M^{G_d} A_{\mathfrak{P}}^{G_i} = J_M^{G_i} A_{\mathfrak{P}}^{G_i}$. Multiplication induces an isomorphism

$$A_{\mathfrak{P}}^{G_i} \otimes_{A_{\mathfrak{P}}^{G_d}} (A_{\mathfrak{P}} \otimes_k M)^{G_d} \simeq (A_{\mathfrak{P}} \otimes_k M)^{G_i},$$

and $A_{\mathfrak{P}}^{G_d} \subset A_{\mathfrak{P}}^{G_i}$ is a free extension. So from the remarks above, it follows that

$$A_{\mathfrak{P}}^{G_i} \otimes_{A^{G_i}} J_M^{G_i} = A_{\mathfrak{P}}^{G_i} \otimes_{A_{\mathfrak{P}}^{G_d}} A_{\mathfrak{P}}^{G_d} \otimes_{A^{G_d}} J_M^{G_d},$$

or $J_M^{G_d} A_{\mathfrak{P}}^{G_i} = J_M^{G_i} A_{\mathfrak{P}}^{G_i}$.

(2) In the second step, we prove that $J_M^G \widehat{A_{\mathfrak{P}}}^{G_d} = J_M^{G_d} \widehat{A_{\mathfrak{P}}}^{G_d}$. Projection induces isomorphisms $\widehat{A}^G \simeq \widehat{A}^{G_d} e_{\mathfrak{P}} (\simeq \widehat{A_{\mathfrak{P}}}^{G_d})$ and

$$(\widehat{A} \otimes_k M)^G \simeq (\widehat{A} e_{\mathfrak{P}} \otimes_k M)^{G_d} (\simeq (\widehat{A_{\mathfrak{P}}} \otimes_k M)^{G_d}),$$

with inverse $\mathrm{Tr}_{G_d}^G$. In particular, if $\omega \in \left(\widehat{A} \otimes_k M\right)^G$, then $e_{\mathfrak{P}}\omega \in \left(\widehat{A} \otimes_k M\right)^{G_d}$ and $\omega = \mathrm{Tr}_{G_d}^G(e_{\mathfrak{P}}\omega)$.

Consider $\omega_1 \wedge \ldots \wedge \omega_m \in \wedge^m_{\widehat{A}} \left(\widehat{A} \otimes_k M\right)$, where $\omega_1, \ldots, \omega_m$ are elements of $\left(\widehat{A} \otimes_k M\right)^G$. From the orthogonality of the idempotents, it follows

$$\begin{aligned} \omega_1 \wedge \ldots \wedge \omega_m &= (\mathrm{Tr}_{G_d}^G(e_{\mathfrak{P}}\omega_1)) \wedge \ldots \wedge (\mathrm{Tr}_{G_d}^G(e_{\mathfrak{P}}\omega_m)) \\ &= \mathrm{Tr}_{G_d}^G \left(e_{\mathfrak{P}}\omega_1 \wedge \ldots \wedge e_{\mathfrak{P}}\omega_m \right), \end{aligned}$$

or, by definition of the Jacobian determinant (with $v_1, \ldots, v_m$ the fixed basis of M),

$$\begin{aligned} \mathrm{Jac}_M (\omega_1, \ldots, \omega_m) \otimes (v_1 \wedge \ldots \wedge v_m) &= \mathrm{Tr}_{G_d}^G \left(\mathrm{Jac}_M (e_{\mathfrak{P}}\omega_1, \ldots, e_{\mathfrak{P}}\omega_m) \otimes (v_1 \wedge \ldots \wedge v_m) \right) \\ &= \mathrm{Tr}_{G_d}^G \left(e_{\mathfrak{P}} \, \mathrm{Jac}_M (\omega_1, \ldots, \omega_m) \otimes (v_1 \wedge \ldots \wedge v_m) \right) \end{aligned}$$

in $\widehat{A}\otimes_k\wedge^m M$. Therefore, $\mathrm{Jac}_M\left(\omega_1\wedge\ldots\wedge\omega_m\right)e_{\mathfrak{P}}=\mathrm{Jac}_M\left(e_{\mathfrak{P}}\omega_1\wedge\ldots\wedge e_{\mathfrak{P}}\omega_m\right)$. We conclude that $J_M^G\widehat{A}^G e_{\mathfrak{P}}$ equals the image of the Jacobian map

$$\wedge^m_{\widehat{A}^{G_d}}\left(\widehat{A}e_{\mathfrak{P}}\otimes_k M\right)^{G_d}\to\widehat{A}e_{\mathfrak{P}},$$

i.e.,

$$J_M^G\widehat{A}^G e_{\mathfrak{P}}=J_M^{G_d}\widehat{A}^{G_d}e_{\mathfrak{P}}=J_M^{G_d}\widehat{A}^{G_d}e_{\mathfrak{P}}.$$

So indeed, $J_M^G\widehat{A_{\mathfrak{P}}}^{G_d}=J_M^{G_d}\widehat{A_{\mathfrak{P}}}^{G_d}$.

(3) In this step, we prove that if $\mathfrak{P}$ contains J_M^G, then the inertia subgroup of $\mathfrak{P}$ is non-trivial. Since $J_M^G\subseteq\mathfrak{P}$, it follows from the second step that $J_M^{G_d}\widehat{A_{\mathfrak{P}}}^{G_d}=J_M^G\widehat{A_{\mathfrak{P}}}^{G_d}\subseteq\mathfrak{P}\widehat{A_{\mathfrak{P}}}^{G_d}$. Since, by the first step, $J_M^{G_i}A_{\mathfrak{P}}^{G_i}=J_M^{G_d}A_{\mathfrak{P}}^{G_i}$, we obtain

$$J_M^{G_i}\widehat{A_{\mathfrak{P}}}^{G_i}=J_M^{G_d}\widehat{A_{\mathfrak{P}}}^{G_i}=J_M^G\widehat{A_{\mathfrak{P}}}^{G_i}\subseteq\mathfrak{P}\widehat{A_{\mathfrak{P}}}^{G_i}.$$

Supposing now that the inertia subgroup is trivial, then $J_M^{G_i}=A$ and we conclude that $\widehat{A_{\mathfrak{P}}}\subseteq\mathfrak{P}\widehat{A_{\mathfrak{P}}}$, which is a contradiction. So G_i is non-trivial.

(4) In this step, we prove (i) and (ii). Let $F_{G,M}$ be a greatest common divisor of all elements on J_M^G. It is unique up to a scalar and $k[V]F_{G,M}$ is the intersection of all height one prime ideals in $k[V]$ containing J_M^G, and so $J_M^G\subseteq k[V]F_{G,M}$. Let f be an irreducible factor of $F_{G,M}$ generating the height one prime ideal $\mathfrak{P}$. The multiplicity μ of f in $F_{G,M}$ then coincides with the integer μ such that $J_M^G A_{\mathfrak{P}}=(\mathfrak{P}A_{\mathfrak{P}})^\mu$ in the discrete valuation ring $A_{\mathfrak{P}}$. It also coincides with the integer μ such that $J_M^G\widehat{A_{\mathfrak{P}}}=(\mathfrak{P}\widehat{A_{\mathfrak{P}}})^\mu$ in the discrete valuation ring $\widehat{A_{\mathfrak{P}}}$. Since we showed in the first two steps that $J_M^G\widehat{A_{\mathfrak{P}}}=J_M^{G_i}\widehat{A_{\mathfrak{P}}}$, the multiplicity of f in $F_{G,M}$ equals the multiplicity of f in $F_{G_i,M}$.

Since $J_M^G\subseteq\mathfrak{P}$, the inertia subgroup of $\mathfrak{P}$ is non-trivial by the third step. Since we are dealing with a linear action, this forces f to be a linear form, say $f=x_U$, where x_U defines a linear subspace $U\subset V$ of codimension one. Then G_i identifies with the point-stabilizer G_U of U, consisting of reflections having U as reflection hyperplan. By Proposition 7, $k[V]^{G_U}(M)$ is free, say with basis $\omega_1,\ldots,\omega_m$. Then $J_M^{G_U}$ is generated by $F_{G_U,M}=\mathrm{Jac}_M(\omega_1,\ldots,\omega_m)$, having necessarily degree $e_U(M)$ (the sum of the degrees of the ω_i's). Since the intersection of G_i with the inertia subgroup of any other height one prime ideal is trivial, it follows that x_U is the only irreducible factor of h, hence up to a scalar we get $F_{G_U,M}=x_U^{e_U(M)}$.

We conclude that $e_U(M)$ is also the multiplicity of x_U in $F_{G,M}$. Comparing with the definition of F_M given before the statement of the theorem, we conclude that $F_M=F_{G,M}$ up to a non-zero scalar. This shows (i) and (ii).

(5) Finally, we prove (iii). Let $\omega_1,\ldots,\omega_m\in([k[V]\otimes_k M)^G$. If (a) holds, then J_M^G is generated by $\mathrm{Jac}_M(\omega_1,\ldots,\omega_m)$, and so by (ii), is equal to F_M up to a non-zero scalar. Hence (b). We remark that $s_{k[V]^G}(k[V]^G(M))$ is equal to the degree of F_M, hence (b) and (c) are clearly equivalent.

Suppose (b). Then $F_M \in J_M^G$ and in particular, $F_M \in k[V]^G_\lambda$. Let $h \in J_M^G$. Then $h \in k[V]^G_\lambda$ and by (ii) there exists a $b \in k[V]$ such that $h = bF_M$ and so $b \in k[V]^G$, i.e. $J_M^G = k[V]^G F_M$.

A square matrix with coefficients in a field is invertible *if and only if* its determinant is non-zero. Hence, if $v_1, \ldots, v_m$ are vectors in an m-dimensional vector space, they form a basis *if and only if* $v_1 \wedge \ldots \wedge v_m$ is non-zero in the top exterior power of the vector space. Let now L be the quotient field of $k[V]$. Since $\mathrm{Jac}_M(\omega_1, \ldots, \omega_m)$ is non-zero, the $\omega_1, \ldots, \omega_m$ form a basis over L^G of $(L \otimes_k M)^G$. Let $\omega \in k[V]^G(M)$ be non-zero. Then it follows that there are $b, b_1, \ldots, b_m \in k[V]^G$ such that

$$b\omega = \sum_i b_i \omega_i.$$

There are $b_i' \in k[V]^G$ such that

$$\mathrm{Jac}_M(\omega_1, \ldots, \omega_{i-1}, b\omega, \omega_{i+1}, \ldots, \omega_m) = bb_i' F_M.$$

On the other hand, using $b\omega = \sum_i b_i \omega_i$, we get

$$b_i F_M = bb_i' F_M.$$

So $b_i = bb_i'$ and $b\omega = b \sum_i b_i' \omega_i$. Since $k[V]^G(M)$ is torsion free, we get

$$\omega = \sum_i b_i' \omega_i,$$

and so $\omega_1, \ldots, \omega_m$ generate the module of covariants of rank m, hence (a). This finishes the proof of (iii). $\square$

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